مجلة جامعة بني وليد للعلوم الإنسانية والتطبيقية
Bani Waleed University Journal of Humanities and Applied Sciences
تصدر عن– جامعة بني وليد – ليبيا
Website:https://jhas-bwu.com/index.php/bwjhas/index
المجلد العاشر ـ العدد الثالث ـ 2025 ـ الصفحات (24 -31)
ISSN3005-3900

Generalized Hyperconnected Sets in Topological Spaces

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المجموعات الشديدة الترابط المعممة في الفضاءات التبولوجية

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تاريخ الاستلام: 2025-05-18 تاريخ القبول: 2025-06-23 تاريخ النشر: 2025-07-01

Abstract
This paper introduces the notion of generalized hyperconnected sets in topological spaces using Levine's concept of generalized closed sets. We define generalized hyperconnected sets and generalized hyperconnected components and study their fundamental properties. Furthermore, we provide necessary conditions for a topological space to be generalized hyperconnected. The relationship between this type of connectedness and other types is also presented. Several results concerning these new concepts are obtained and proven.

Keywords: Hyper connected set, g-closed set, gh-component, g-hyperconnected set, g-open set, $T_{\frac{1}{2}}$ space.

الملخص
في هذا البحث، تم تقديم مفهوم المجموعات الشديدة الترابط المعممة في الفضاءات التبولوجية و ذلك باستخدام مفهومي المجموعات المفتوحة المعممة والمجموعات المغلقة المعممة، كما تم تقديم مفهوم المركبات الشديدة الترابط المعممة . كذلك، تمت دراسة الفضاءات التبولوجية الشديدة الترابط المعممة، وتحديد بعض الشروط الضرورية التي يجب أن تتحقق ليكون الفضاء التبولوجي شديد الترابط المعمم. و قد تم التوصل إلى عدد من النتائج ، كما تمت دراسة العديد من الخصائص المرتبطة بهذه المفاهيم.

الكلمات الدالة: فضاء $T_{\frac{1}{2}}$، مجموعة شديدة الترابط، مجموعة شديدة الترابط المعممة، مجموعة مغلقة معممة، مجموعة مفتوحة معممة، مركبة شديدة الترابط المعممة .

Introduction

The notion of generalized closed sets in topological spaces (or g-closed) was introduced and studied by Levine (1970). Many topological concepts related to g-closed sets have been studied extensively by many topologists. Balachandran et al. (1991) defined and studied generalized continuous maps, while Dunham (1982) introduced the notion of a g-closure operator. Several g-topological concepts have been defined and developed (see, for instance, Caldas et al., 2007; Cao et al., 2002; Dunham, 1977). The concept of hyperconnected spaces was introduced by Steen and Seebach (1970), and various related concepts have been investigated by many researchers (see for instance, Chutiman & Boonpok, 2023; Sasikala & Deepa, 2021).

The purpose of this paper is to introduce and investigate the notion of g-hyperconnected sets in topological space. We study various properties of g-hyperconnected sets and prove some basic results. Furthermore, we introduce and study the notion of gh-components. Finally, we provide necessary conditions for g-hyperconnected topological spaces and investigate some of their properties.

2. Preliminaries

Definition 1. A subset F of a topological space X is said to be generalized closed (briefly, g-closed) if $\bar{F} \subseteq V$ whenever V is open and $F \subseteq V$. The complement of a g-closed set is called generalized open (briefly, g-open) (Levine, 1970).

The class of all g-closed (resp., g-open) sets will be denoted by $GF(X)$ (resp., $GO(X)$).

Observation 1. (1) Every closed set is g-closed (Levine, 1970).

(2) If F_1 and F_2 are g-closed sets, then $F_1 \cup F_2$ is g-closed, while $F_1 \cap F_2$ need not be g-closed (Levine, 1970).

Definition 2. A topological space is $T_{\frac{1}{2}}$ if each g-closed set is closed (Levine, 1970).

Theorem 1. Every T_1 space is $T_{\frac{1}{2}}$ and every $T_{\frac{1}{2}}$ space is T_0 (Levine, 1970).

Theorem 2. A topological space is $T_{\frac{1}{2}}$ if and only if each singleton is either closed or open (Dunham, 1977).

Theorem 3. For each $y \in X$, either $\{y\}$ is closed or the complement of $\{y\}$ is g-closed (Dunham, 1982).

Definition 3. The g-closure $\bar{B}^g$ of a subset B of X is the intersection of all g-closed sets that contains B (Dunham, 1982).

Definition 4. A topological space X is said to be hyperconnected (resp., ultraconnected) if X cannot be written as the union of two proper closed (resp., open) sets (Steen & Seebach, 1970).

Definition 5. A topological space is said to be g-connected if it cannot be written as a disjoint union of two nonempty g-open sets (Balachandran et al., 1991).

Definition 6. A function $h: X \rightarrow Y$ is said to be g-irresolute (Balachandran et al., 1991) (resp., g-continuous (Balachandran et al., 1991), contra g-continuous (Caldas et al., 2007)) if the inverse image of every g-closed (resp., closed, open) set in Y is g-closed (resp., g-closed, g-closed) in X.

3. g-Hyperconnected Sets in Topological Spaces

In this section, we introduce and investigate the concept of g-hyperconnected sets in topological spaces.

Definition 7. A subset A of X is said to be g-hyperconnected if $A \subseteq F$ or $A \subseteq G$ whenever $A \subseteq F \cup G$, F, G are g-closed sets.

Example 1. Let $X = \mathbb{N}$ with the cofinite topology, then any infinite subset of X is g-hyperconnected.

Theorem 4. Let B be a g-hypeconnected set in X and $A \subseteq B$. If A is g-open in X, then A is g-hyperconnected.

Proof. Suppose, for a contradiction, that A is not g-hyperconnected in X, so there are g-closed sets F and G such that $A \subseteq F \cup G$ and $A \nsubseteq F$, $A \nsubseteq G$. Define $H = (X \backslash A) \cup G$, so H is g-closed since both $X \backslash A$ and G are g-closed. Observe that $B \subseteq F \cup H$ and $B \nsubseteq F$, $B \nsubseteq H$, which contradicts the assumption that B is a g-hyperconnected set. Therefore, A is a g-hyperconnected set in X.

Corollary 1. The interior of a g-hyperconnected set is also g-hyperconnected.

Observation 2. If A is a subset of a g-hyperconnected set B and A is g-closed in X, then A need not be g-hyperconnected. To illustrate this, consider the following example.

Example 2. Let τ_p be the particular point topology on a set X, where $|X| > 2$, then X is g-hyperconnected. If $A \subsetneq X$, $|A| > 1$, and $p \notin A$, then A is g-closed but not g-hyperconnected.

The following theorem shows that a g-hyperconnected set has a g-hyperconnected g-closure.

Theorem 5. If A is a g-hyperconnected set in X, then $\bar{A}^g$ is g-hyperconnected.

Proof. Suppose, for a contradiction, that $\bar{A}^g$ is not g-hyperconnected, so there are g-closed sets F, G such that $\bar{A}^g \subseteq F \cup G$, and $\bar{A}^g \nsubseteq F$, $\bar{A}^g \nsubseteq G$. Since $A \subseteq \bar{A}^g$, which implies that $A \subseteq F \cup G$, we have two cases:

(1) If $A \nsubseteq F$ and $A \nsubseteq G$, then A is not g-hyperconnected, which is a contradiction.

(2) If $A \subseteq F$ (the case $A \subseteq G$ is similar), then by the definition of the g-closure and the assumption that F is g-closed, we have $\bar{A}^g \subseteq F$, which is a contradiction of our assumption that $\bar{A}^g \nsubseteq F$.

Therefore, from cases (1) and (2), we conclude that $\bar{A}^g$ is g-hyperconnected in X.

Observation 3. If $A_1, A_2 \subseteq X$ are g-hyperconnected sets and $A_1 \cap A_2 \neq \phi$, it is not true in general that $A_1 \cup A_2$ is g-hyperconnected; this is illustrated by the following example.

Example 3. Let $X = \{a, b, d\}$ and $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$, so we have $GF(X) = \{\phi, X, \{b, d\}, \{a, d\}, \{d\}\}$. If $A_1 = \{a, d\}$ and $A_2 = \{b, d\}$, then $A_1 \cap A_2 \neq \phi$. In this case, each of A_1 and A_2 is g-hyperconnected, but $A_1 \cup A_2 = X$ is not g-hyperconnected.

The following theorem describes the images of g-hyperconnected sets under certain types of functions.

Theorem 6. Let $h: X \to Y$ be a surjection and let B be g-hyperconnected in X.

(1) If h is g-irresolute then $h(B)$ is g-hyperconnected in Y.

(2) If h is g-continuous then $h(B)$ is hyperconnected in Y.

Proof. (1) Suppose, for a contradiction, that $h(B)$ is not g-hyperconnected in Y, so $h(B) \subseteq F \cup G$ for some g-closed sets F and G, and $h(B) \nsubseteq F$, $h(B) \nsubseteq G$. Now, both $h^{-1}(F)$ and $h^{-1}(G)$ are g-closed in X since h is g-irresolute. It follows that $B \subseteq h^{-1}(F) \cup h^{-1}(G)$ with $B \nsubseteq h^{-1}(F)$ and $B \nsubseteq h^{-1}(G)$, so B is not g-hyperconnected in X, which is a contradiction.

(2) The proof of this statement follows along the same lines as in (1).

Now, we give the definition of gh-components in a topological space.

Definition 8. A g-hyperconnected component (briefly, gh-component) in X is defined to be a maximal g-hyperconnected set in X.

Example 4. Let (X, τ_p) be an excluded point space, then for each $x \in X \backslash \{p\}$, $\{p, x\}$ is a gh-component in X.

Observation 4. (1) If C and C^* are gh-components, then it is not necessarily true that $C \cap C^* = \phi$. In Example 4, the intersection of any two gh-components is $\{p\}$.

(2) The union of all gh-components in X is X.

A gh-component is not necessarily closed, but the following result shows that each gh-component in a space is equal to its g-closure.

Theorem 7. If C is a gh-component in X then $C = \bar{C}^g$.

Proof. Since C is a g-hyperconnected set, it follows from Theorem 5 that $\bar{C}^g$ is also g-hyperconnected in X. By the definition of the g-closure we have $C \subseteq \bar{C}^g$. Since C is a gh-component, i.e., a maximal g-hyperconnected set, it follows that $\bar{C}^g \subseteq C$. Therefore, $C = \bar{C}^g$.

Theorem 8. If $h: X \to Y$ is a g-irresolute surjection and if C is a gh-component in X then $h(C)$ must lie in a gh-component in Y.

Proof. Since C is g-hyperconnected in X, then, from Theorem 6, we have $h(C)$ is also g-hyperconnected in Y. It follows immediately that $h(C) \subseteq C^*$ for some gh-component C^* in Y.

4. g-Hyperconnected Topological Spaces

Definition 9. X is said to be a g-hyperconnected space if X cannot be written as the union of two proper g-closed subsets of X. Equivalently, X is g-hyperconnected if any two nonempty g-open sets intersect.

Obviously, any g-hyperconnected space is hyperconnected, but the converse is generally false as shown by:

Example 5. Let X be an indiscrete space with $|X| > 2$, so $GF(X)$ is the power set of X. Hence, X is hyperconnected but not g-hyperconnected.

The implications between g-hyperconnectedness and different types of connectedness can be summarized by:

g-hyperconnected space	$\Longrightarrow$	hyperconnected space
$\Downarrow$		$\Downarrow$
g-connected space	$\Longrightarrow$	connected space

Observation 5. It is known that if $\tau_1 \subseteq \tau_2$ and (X, τ_2) is hyperconnected (resp., connected), then (X, τ_1) is also hyperconnected (resp., connected). However, if (X, τ_2) is g-hyperconnected then (X, τ_1) need not be g-hyperconnected. For example, if (X, τ_2) is any g-hyperconnected space and (X, τ_1) is the indiscrete space, then $GF_1(X)$ is the power set of X, so (X, τ_1) is not g-hyperconnected.

The following theorem provides necessary (but not sufficient) conditions that a space must satisfy to be g-hyperconnected.

Theorem 9. If X is a g-hyperconnected space, then either X is a $T_{1/2}$ space or X satisfies the following condition: there exists $y \in X$ such that $\{y\}$ is neither closed nor open, and for each $x \neq y$, $\{x\}$ is closed and not open.

Proof. Suppose X is neither a $T_{1/2}$ space nor satisfies the condition above. This assumption, together with Theorem 2, implies that we have exactly the following possible cases:

(1) There are $x, y \in X$, $x \neq y$, such that each of $\{x\}$ and $\{y\}$ is neither closed nor open. Hence, from Theorem 3, we have $X \backslash \{x\}$ and $X \backslash \{y\}$ are g-closed sets. Therefore, since $X = X \backslash \{x\} \cup X \backslash \{y\}$, X is not g-hyperconnected.

(2) There are $x, y \in X, x \neq y$, such that $\{y\}$ is open, and $\{x\}$ is neither closed nor open. Thus, $X \backslash \{y\}$ is a closed set, and from Theorem 3, $X \backslash \{x\}$ is g-closed. Since $X = X \backslash \{x\} \cup X \backslash \{y\}$, X is not g-hyperconnected.

Therefore, from cases (1) and (2), the proof is complete.

Observation 6. The converse of Theorem 9 is generally false; we illustrate this by the following examples.

Example 6. $\mathbb{R}$ with the usual topology is a $T_{1/2}$ space, but it is not g-hyperconnected.

Example 7. Let $X = \mathbb{N}$ and let τ be the topology defined as follows

$$\tau = \{\phi, X \backslash C : C \text{ is finite and either } 1 \notin C \text{ or } \{1,2\} \subseteq C\},$$

so each singleton in X is closed and not open except the singleton $\{1\}$, which is neither closed nor open. According to Theorem 3, the set $X \backslash \{1\}$ is g-closed in X. Therefore, since $X = (X \backslash \{1\}) \cup \{1,2\}$, X is not g-hyperconnected.

From Theorem 1 and Theorem 9, we have the following two corollaries.

Corollary 2. If X is a $T_{1/2}$ space, then X is hyperconnected if and only if X is g-hyperconnected.

Corollary 3. If X is a g-hyperconnected space, then:

(1) X is a T_0 space.

(2) Any two distinct points in X cannot be separated by disjoint g-open sets, so X is not a Hausdorff space. Moreover, X is neither a regular space nor a normal space.

(3) No g-continuous function $h: X \to \{0,1\}$ is surjective, where $\{0,1\}$ is the discrete space.

Observation 7. Let $A \subseteq X$, and suppose that (A, τ_A) is g-hyperconnected as a topological subspace of X, then A need not be g-hyperconnected as a subset of X. This can be illustrated by the following example.

Example 8. Let $\mathbb{N}$ be equipped with the cofinite topology, $Y = \{0,1\}$ be the indiscrete space, and $X = \mathbb{N} \times Y$ be the product space (also known as the double-pointed cofinite space) (Steen & Seebach, 1970). If $A = \mathbb{N} \times \{0\}$, then the subspace (A, τ_A) has the cofinite topology and is therefore g-hyperconnected. However, A is not g-hyperconnected as a subset of X since, from Theorem 3, $X \backslash \{(y, 0)\}$ is g-closed for each $y \in \mathbb{N}$; so $A \subseteq X \backslash \{(a, 0)\} \cup X \backslash \{(b, 0)\}$ for some distinct points $a, b \in \mathbb{N}$, and $A \nsubseteq X \backslash \{(a, 0)\}$, $A \nsubseteq X \backslash \{(b, 0)\}$.

Theorem 6 showed that g-hyperconnectedness is preserved under g-irresolute surjections. Now, we consider a different class of functions and obtain the following result.

Theorem 10. Let $f: X \to Y$ be a contra g-continuous surjection. If X is g-hyperconnected, then Y is ultraconnected.

Proof. If Y is not ultraconnected, then $Y = U \cup W$ for some proper open sets U, V; so $X = f^{-1}(U) \cup f^{-1}(W)$. But $f^{-1}(U)$, $f^{-1}(W)$ are g-closed in X since f is contra g-continuous, so X is not g-hyperconnected.

5. Conclusion

In this research, the concept of generalized hyperconnected sets was introduced and studied using g-open and g-closed sets. Other forms of hyperconnectedness can be investigated by using different types of generalized open sets, such as pre-g-open sets and semi-g-open sets. Moreover, this concept can also be extended to other branches of topology, such as soft topology.

Although this study has focused on the purely theoretical aspects of general topology, it is important to note that topological concepts have also been applied in various practical fields such as data analysis and decision-making problems (see, for example, Patel & Duraphe, 2024; Chazal & Michel, 2021). Applying generalized forms of connectedness in such fields is likely to be useful for understanding the relationships and connectivity among specific elements within a system, and it can lead to valuable results and improved decision-making processes. Future research related to generalized hyperconnectedness could be pursued in this direction and other practical fields.

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