المؤتمر العلمي الدولي الأول للهندسة و العلوم
The First Scientific International Conference in Engineering & Science

http://bwu.edu.ly/icse2022
Received 30/07/2022 Revised 30/12/2022 Published 05/01/2023
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A Cubic B-Spline Collocation Method for Solving Anomalous Sub-diffusion Equation

Faoziya S. Musbah[a]
[a]Department of Mathematics, Faculty of education/University of Bani Waleed, Libya
*Crosspnding author: faoziya_sh@yahoo.com

Abstract: In this paper, A cubic B-spline collocation method is proposed to solve one dimensional anomalous sub-diffusion equation. The fractional derivative is estimated by using right shifted Grünwald-Letnikov formula of order $\alpha \in (0,1)$. Numerical results are presented to confirm the feasibility and validity of this scheme.

Keywords: A Cubic B-spline collocation method, anomalous sub-diffusion equation, Grünwald-Letnikov formula.

Introduction

Paper must Diffusion equations are partial differential equations which model the diffusion and thermodynamic phenomena and describe the spread of particles (ions, molecules, etc.) diffusion not described by normal diffusion in the long time limit has become known as anomalous (unnatural) [1]. Fractional partial differential equations can be thought as generalizations of classical partial differential equations, which can give a better description of the complex phenomena such as signal processing, systems identification, control and non-Brownian motion [2] or so called levy motion which is a generalization of Brownian motion [3]. A comprehensive background on this topic can be found in books by [4] and [5].

In this article, a numerical scheme is constructed to obtain approximate solutions of the one-dimensional anomalous sub-diffusion equation. The Grünwald-Letnikov formula is applied to treat the fractional temporal derivative, while the cubic B-spline (CBS) is used to discretize the spatial derivative.
Consider the following model of anomalous sub-diffusion equation:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = D\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad (x,t) \in (0,1)\times(0,T], \tag{1}$$

with the initial condition

$$u(x,0) = g_1(x), \quad x \in [0,1], \tag{2}$$

and the boundary conditions

$$u(0,t) = g_2(t), \quad u(1,t) = g_3(t), \quad t \in [0, T], \tag{3}$$

where $u(x,t)$ is a concentration of a quantity such as mass, energy, etc., D is the diffusion coefficient (or diffusivity), f, g_1, g_2 and g_3 are known functions. $\partial^\alpha u / \partial t^\alpha$ denotes the Riemann-Liouville fractional derivative. We consider the case when $0 < \alpha < 1$.

Definition 1.1 The fractional derivative ${}_0D_t^\alpha$ of $f(t)$ can be defined by Riemann-Liouville formula as [6]

$${}_0D_t^\alpha f(t) = \frac{d}{dt}\left[\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f(\iota)d\iota}{(t-\iota)^\alpha}\right], \quad 0 < \alpha < 1, \tag{4}$$

where $\Gamma(.)$ is the Gamma function and $0 \le t \le T$. The above derivative is related to the Riemann-Liouville fractional integral, which is defined as

$${}_0I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(\iota)d\iota}{(t-\iota)^{1-\alpha}}, \quad 0 < \alpha < 1, \tag{5}$$

Where ${}_0D_t^\alpha\, {}_0I_t^\alpha f(t) = f(t)$.

Definition 1.2 The right-shifted Grünwald-Letnikov formula of function f with respect to independent variable t is defined as [7]

$$ {}_0D_t^{\alpha} f(t) = \frac{1}{\tau^{\alpha}} \sum_{k=0}^{n+1} \omega_k^{(\alpha)} f(t-(k-1)\tau) + O(\tau) \qquad (7)$$

where $\tau = \Delta t$, $\omega_0^{(\alpha)} = 1$ and

$\omega_k^{(\alpha)} = (1 - \frac{\alpha+1}{k})\omega_{k-1}^{(\alpha)}$.

The coefficients $\omega_k^{(\alpha)}$ are the coefficients of the power series of the generating function $\omega(z,\alpha) = (1-z)^{\alpha}$ and are also the coefficients of the two-point backward difference approximation of the first order derivative. The generating function $\omega(z,\alpha)$ with $0 < \alpha < 1$ can be written as a power series of the form

$$(1-z)^{\alpha} = \sum_{k=0}^{\infty} \binom{k-\alpha-1}{k} z^k = \sum_{k=0}^{\infty} \omega_k^{(\alpha)} z^k \qquad (8)$$

Cubic B-spline functions

First, we introduce a uniform grid of mesh points (x_i, t_n) with $x_i = ih, i = 0,1,\dots,M$ and $t_n = n\tau, n = 0,1,\dots,N$, where M and N are positive integers, $h = 1/M$ is the spatial step size in the x direction and $\tau = T/N$ is the time step size in the t direction. The notations u_i^n and f_i^n are used for the exact values of u and f at the points (x_i, t_n). An approximation U to the exact solution u can be expressed in terms of the cubic B-spline collocation approach as [8]

$$U(x,t) = \sum_{i=-1}^{M} C_i(t) B_i(x), \qquad (9)$$

where $C_i(t)$ are unknown control points and $B_i(x)$ are CBS functions defined as

$$B_i(x) = \frac{1}{6}\begin{cases} (x-x_{i-2})^3, & x \in [x_{i-2}, x_{i-1}] \\ h^3 + 3h^2(x-x_{i-1}) + 3h(x-x_{i-1})^2 - 3(x-x_{i-1})^3, & x \in [x_{i-2}, x_{i-1}] \\ h^3 + 3h^2(x_{i+1}-x) + 3h(x_{i+1}-x)^2 - 3(x_{i+1}-x)^3, & x \in [x_{i-2}, x_{i-1}] \\ (x_{i+2}-x)^3, & x \in [x_{i-2}, x_{i-1}] \\ 0, & \text{otherwise.} \end{cases}$$

The value of $B_i(x)$, $B_i'(x)$ and $B_i''(x)$ at mesh point x_i are represented in Table 1.

Table 1: The value of $B_i(x)$, $B_i'(x)$ and $B_i''(x)$ at mesh point x_i.

	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
$B_i(x)$	0	1/6	4/6	1/6	0
$B_i'(x)$	0	$1/2h$	0	$-1/2h$	0
$B_i''(x)$	0	$1/h^2$	$-2/h^2$	$1/h^2$	0

The set of B-spline functions $\{B_{-1}(x), B_0(x), \dots, B_M(x)\}$ is defined over $[a,b]$. Therefore, an approximation solution u_i^n at the point (x_i, t_n) over the subinterval $[x_i, x_{i+1}]$ is given as

$$u_i^n = \sum_{j=i-1}^{i+1} C_j^n B_j(x) \qquad (10)$$

where $i = 1,2,\dots,M-1$. The approximated values of u and $\partial^2 u/\partial x^2$ are computed in term of the control points C_j^n as:

$$u_i^n = \frac{1}{6}\left(C_{i-1}^n + 4C_i^n + C_{i+1}^n\right) \qquad (11)$$

and

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_i^n = \frac{1}{h^2}\left(C_{i-1}^n - 2C_i^n + C_{i+1}^n\right) \qquad (12)$$

Cubic B-spline collocation method for solving anomalous sub-diffusion equation

In this section, the CBS collocation method is constructed for anomalous sub-diffusion equation. The time fractional term of equation (1) is treated by using right shifted Grünwald-Letnikov formula (7) while its second spatial derivatives is replaced by applying the approximation formula (12), taking into the consideration that the estimation of truncation errors of this scheme are neglected. We set

$$\frac{1}{\tau^{\alpha}}\sum_{k=0}^{n}\omega_k^{\alpha}u_i^{n-k}=\frac{D}{h^2}\frac{\partial^2 u}{\partial x^2}\bigg|_i^n+f_i^n,\; i=1,2,\ldots,M-1,$$

$$n=1,2,\ldots,N \tag{13}$$

Substitute equation (11) and (12) into equation (1), yields

$$\omega_0^{\alpha}\left(C_{i-1}^n+4C_i^n+C_{i+1}^n\right)+\sum_{k=1}^{n}\omega_k^{\alpha}\left(C_{i-1}^{n-k}+4C_i^{n-k}+C_{i+1}^{n-k}\right)$$

$$=6r\left(C_{i-1}^n-2C_i^n+C_{i+1}^n\right)+6\tau^{\alpha}f_i^n,\; i=1,2,\ldots,M-1,$$

$$n=1,2,\ldots,N \tag{14}$$

where $r=D\tau^{\alpha}/h^2$ is the fractional diffusion number. Simplifying the above equation, we obtain

$$\left(\omega_0^{\alpha}-6r\right)C_{i-1}^n+4\left(\omega_0^{\alpha}+3r\right)C_i^n+\left(\omega_0^{\alpha}-6r\right)C_{i+1}^n$$

$$=6\tau^{\alpha}f_i^n-\sum_{k=1}^{n}\omega_k^{\alpha}\left(C_{i-1}^{n-k}+4C_i^{n-k}+C_{i+1}^{n-k}\right),\; i=1,2,\ldots,M-1,$$

$$n=1,2,\ldots,N \tag{15}$$

The system in (15) has M-1 linear equations involving M+1 unknowns. The two additional equations can be obtained from the boundary conditions of equation (1).

Since the equation (10) is assumed to be an approximate solution for anomalous sub-diffusion equation (1) then from the boundary conditions (3) we can write

$$u_0^n=\sum_{j=-1}^{1}C_j^nB_j(x)=g_2(t),\;\frac{\partial^2 u}{\partial x^2}\bigg|_0^n=\sum_{j=-1}^{1}C_j^nB_j^{''}(x)=0, \tag{16}$$

$$u_M^n=\sum_{j=M-1}^{M+1}C_j^nB_j(x)=g_3(t),\;\frac{\partial^2 u}{\partial x^2}\bigg|_M^n=\sum_{j=M-1}^{M+1}C_j^nB_j^{''}(x)=0. \tag{17}$$

Hence, this will lead to the following linear system of (M+1) equations and (M+1) unknowns:

$$\underline{A}\underline{C^n}=\underline{b},\quad i=1,2,\ldots,M-1,\; n=1,2,\ldots,N \tag{18}$$

where $\underline{C^n}=[C_0^n,C_1^n,\ldots,C_M^n]^T$. The matrix $\underline{A}$ in (18) is defined as

$$\underline{A}=\begin{pmatrix} 2 & 1 & & & \\ \omega_0^{\alpha}-6r & 4(\omega_0^{\alpha}+3r) & \omega_0^{\alpha}-6r & & \\ & \ddots & \ddots & \ddots & \\ & & \omega_0^{\alpha}-6r & 4(\omega_0^{\alpha}+3r) & \omega_0^{\alpha}-6r \\ & & & 1 & 2 \end{pmatrix}_{(M+1)\times(M+1)}$$

The matrix $\underline{A}$ is tri-diagonally strictly dominant and guaranteed that the solution of the system is unique.

The vector $\underline{b}$ for $n=1,2,\ldots,N$ is in the form

$$\underline{b}=\begin{pmatrix} 6\tau^{\alpha}f_1^n-\sum_{k=1}^{n}\omega_k^{\alpha}\left(C_0^{n-k}+4C_1^{n-k}+C_2^{n-k}\right) \\ \vdots \\ 6\tau^{\alpha}f_{M-2}^n-\sum_{k=1}^{n}\omega_k^{\alpha}\left(C_{M-3}^{n-k}+4C_{M-2}^{n-k}+C_{M-1}^{n-k}\right) \\ 6\tau^{\alpha}f_{M-1}^n-\sum_{k=1}^{n}\omega_k^{\alpha}\left(C_{M-2}^{n-k}+4C_{M-1}^{n-k}+C_{M}^{n-k}\right) \end{pmatrix}$$

Note that, the value of the coefficients $C_0^{n-k},C_1^{n-k},...,C_M^{n-k}$ for $k=1,2,...,N$ are given from the initial value, at time level *n*. After calculating the coefficients $C_i^n(i=0,1,...,M,$ and $i=1,2,...,N)$ the approximate values $u_i^n(i=1,2,...,M-1,$ and $i=1,2,...,N)$ is determined by using the formula (10) to finally obtain the CBS approximation.

Numerical Experiments

In this section, the proposed scheme will be tested to confirm the performance and the effectiveness of the present scheme. The numerical results are carried out using Mathematica wolfram 8.

Example 1. Let us consider the equation from reference [9]

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\frac{\partial^2 u(x,t)}{\partial x^2}+\frac{2e^xt^{2-\alpha}}{\Gamma(3-\alpha)}-t^2e^x,\;(x,t)\in(0,1)\times(0,T], \tag{19}$$

with the initial condition

$$u(x,0)=0,\quad x\in[0,1], \tag{20}$$

and the boundary conditions

$$u(0,t)=t^2, \quad u(1,t)=et^2, \quad t\in[0,\ T], \qquad (21)$$

The exact solution of equation (19)-(21) is

$$u(x,t)=t^2e^x. \qquad (22)$$

We choose $\alpha=0.5$. The numerical scheme discussed in this paper for solving the above example is implemented and its solution is compared with the exact solution. The computational results of Example 1 are illustrated in Table 2 along with the relative errors at $t=6.25\times10^{-5}$, $\tau=1.25\times10^{-5}$ and $h=0.1$.

Table 2: Relative errors of the scheme at $t=6.25\times10^{-5}$, $\tau=1.25\times10^{-5}$ and $h=0.1$.

x	Approx. soln.	Exact soln.	Errors
0.1	4.50256E-9	4.31707E-9	4.29649E-2
0.2	5.19015E-9	4.77110E-9	8.78295E-2
0.3	5.79088E-9	5.27289E-9	9.82365E-2
0.4	6.41295E-9	5.82744E-9	1.00475E-1
0.5	7.08946E-9	6.44032E-9	1.00793E-1
0.6	7.83142E-9	7.11765E-9	1.00282E-1
0.7	8.63749E-9	7.86622E-9	9.80476E-2
0.8	9.47852E-9	8.69352E-9	9.02977E-2
0.9	1.02425E-8	9.60782E-9	6.60633E-2

The comparison between the results of the CBS collocation method and the exact solution is plotted in Fig. 1 at $t=6.25\times10^{-5}$, $\tau=1.25\times10^{-5}$ and $h=0.1$.

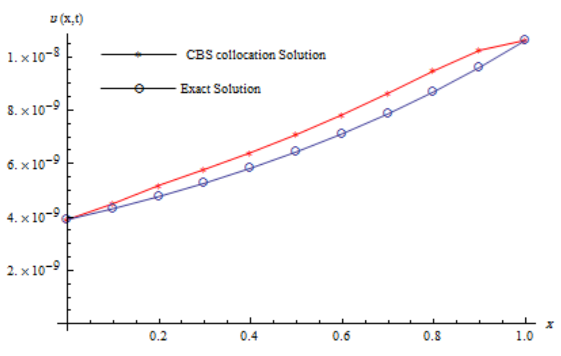

Fig. 1: Comparison between the results of CBS collocation method and exact solution at $t=6.25\times10^{-5}$, $\tau=1.25\times10^{-5}$ and $h=0.1$.

From Fig. 1, we can observe that the numerical solutions of the CBS collocation method are close to the exact solutions. In Fig.2 the relative errors of the numerical CBS collocation method are presented at $t=6.25\times10^{-5}$, $\tau=1.25\times10^{-5}$ and $h=0.1$.

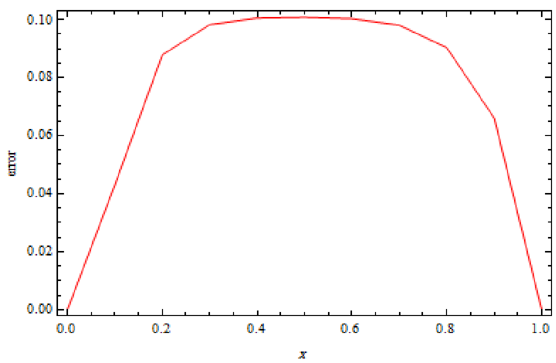

Fig. 2: The relative errors of the scheme at $t=6.25\times10^{-5}$, $\tau=1.25\times10^{-5}$ and $h=0.1$.

Conclusion

In this paper, a numerical scheme based on CBS was presented for solving anomalous sub-diffusion equation. The time fractional derivative was estimated via Grünwald-Letnikov formula while the spatial derivative was utilized using the CBS approximation. The proposed algorithm was tested by a numerical example, which showed that the scheme is admissible, straightforward and produced reasonable results.

Arabic section:

عنوان البحث: طريقة البي سبلاين المنتظمة لحل معادلة الانتشار ذات الاشتقاق الكسري.

الإسم: فوزية صالح محمود مصباح.

ملخص البحث: في هذا البحث تم اقتراح طريقة البي سبلاين المنتظمة لحل معادلة الانتشار ذات الاشتقاق الكسري. المشتقة الكسرية تم معالجتها باستخدام صيغة الـ Grünwald-Letnikov من الرتبة $\alpha\in(0,1)$ النتائج العددية للطريقة المقترحة قد تم عرضها لإثبات مدى صلاحية وفاعلية هذه الطريقة.

الكلمات المفتاحية: طريقة البي سبلاين المنتظمة، معادلة الانتشار ذات الاشتقاق الكسري، Grünwald-Letnikov.

Abbreviations and Acronyms

CBS (Cubic B-spline).

Approx. (Approximate)

Soln. (Solution)

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