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طريقة ستيفنسن لحل معادلات فريدهولم التكاملية من النوع الثاني

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Steffensen’s Method for Solving Fredholm Integral Equations of the Second Kind

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الملخص:

في هذه الورقة تم عرض طريقة ستيفنسن الحسابية لحل معادلة فريدهولم التكاملية من النوع الثاني، ويعتمد المخطط على صيغ التربيع المفتوحة، وتم إجراء مقارنة بين نتائج الحل الدقيق والحل التقريبي وأظهرت أن المخطط واضح ومباشر وقابل للتطبيق على هذه المعادلة التكاملية وأنواع أخرى من المعادلات التكاملية.

الكلمات المفتاحية:طريقة ستيفنسن، معادلة فريدهولم التكاملية، صيغ بيع مفتوحة، الحل الدقيق، الحل التقريبي.

Abstract:

In this article, the computational Steffensen’s method is presented for solving Fredholm integral equation of the second kind. The scheme is based on open quadrature formulas. A comparison between the results of the exact solution and the approximate solution were done and showed that the scheme is straightforward and applicable for this integral equation and other types of integral equations.

Keywords: Stevenson's method, Fredholm integral equation, open sale formulas, exact solution, approximate solution.

1. Introduction

The most general type of linear integral equation in $y(x)$ is of the form [1]:

$$g(x)y(x) = f(x) + \lambda \int_{a}^{*} K(x,t)y(t)dt \qquad (1)$$

The upper limit (*) may be either variable or fixed. The functions g, f and the so-called kernel k are given, while the function $y(x)$ is unknown, λ is a constant and non-zero parameter and its value is called eigenvalue, or characteristic value of that equation [2]. If the limits of integration in (1) are constant, the integral equation is called a Fredholm integral equation, such an equation can be derived from boundary value problems with given boundary conditions [1]. FIEs have numerous applications in various fields, including potential theory, wave propagation, scattering theory, heat transfer, elasticity, and fluid flow [3]. Yang Y., Tang Z., and Huang T. Y. [4] have introduced a numerical scheme based on collocation method to solve Fredholm integral equation of the second kind with weakly singular kernel. The Jacobi-Gauss quadrature formula is used to approximate the integral part of the equation. The errors of this method were also obtained and they were exponentially decayed in $L^{\infty} -$ norm. Hamedzadeh D., and Babolian E. [5] have studied Taylor series of unknown function to remove the singularity to solve computationally Fredholm integral equation of the second kind with weakly singular kernel. Shoukralla, E. S., et. al. [6] have considered Legendre polynomials in matrix form to solve a linear system of unknown functions to obtain approximate solution of Fredholm integral equation of the second kind with weakly singular kernel. The convergence of this method was also proved.

The purpose of this article is to consider a computational Steffensen's method to solve Fredholm integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt \tag{2}$$

where a and b are constants, $a \le x \le b$.

2. Steffensen's Method

In this section, a computational Steffensen's method is constructed for solving Fredholm integral equation of the second kind. The discrete point is $x_i = a + ih$, $i = 1,2,\dots,n-1$ where n is a positive integer, $h = \frac{b-a}{n+2}$ is the step size in the x direction, $f_i = f(x_i)$ and $K_{ij} = K\left(x_i, t_j\right)$.

The open quadrature formulas are given in the following form [7]:

$$\int_a^b p_n(x)dx = h\int_{-1}^{n+1}\sum_{j=0}^{n}\Delta^j f_0\binom{s}{j}ds, \qquad \binom{s}{j} = \frac{s!}{j!\,(s-j)!}, \qquad (3)$$

where $\Delta^j f_0 = \Delta(\Delta^{j-1} f_0)$ is called forward differences of the order j, and

$$s = \frac{x - x_0}{h}, \qquad h = \frac{b-a}{n+2}, \qquad x_0 = a + h, \qquad x_n = b - h, \;\; x_i = x_0 + ih$$

The formula corresponding to $n = 3$ is called Steffensen's formula [8] and can be given as follows:

$$\int_a^b f(x)dx \cong \frac{5h}{24}\big(11f(x_0) + f(x_1) + f(x_2) + 11f(x_3)\big) \qquad (4)$$

The truncation error term E_S of this formula is

$$E_S = \frac{95h^5}{144} f^4(x_0 + \xi h), \qquad \xi \in [-1,4].$$

This method has a degree of precision $n = 5$ and the truncation error is of the order $O(h^5)$. Generally, the degree of precision of the n-point Newton-cotes quadrature formula is $(n-1)$ when n is even and n when n is odd [9].

The solution of the Fredholm integral equation (2) can be approximated by using Steffensen's formula (4) at $u_i = u(x_i)$ as

$$y_i = f_i + \frac{h}{2}[K_{i0}y_0 + 2(K_{i1}y_1 + K_{i2}y_2 + \cdots + K_{ii-1}y_{i-1}) + K_{ii}y_i],$$
$$i = 1,2,\ldots,n-1 \qquad (5)$$

3. Numerical Examples

In this section, numerical examples are presented to tested to measure the accuracy of proposed method. The numerical results are carried out using Mathematica wolfram 8.

Example 1. Consider the following Fredholm integral of the second kind equation [10]

$$y(x) = -\frac{\pi}{2}\cos x + \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos(x-t)y(t)dt, \qquad x \in \left[0, \frac{\pi}{2}\right] \tag{6}$$

The exact solution of Eq. (6) is $y(x) = \sin x$.

The computational Steffensen's scheme introduced in this article for solving Fredholm integral equation is implemented and its solution is compared with the exact solution. The numerical solution of Example 1 is illustrated in Table 3.1 along with the absolute errors at $= 3$, and $h = \pi/10$.

Table 3. 1. Absolute errors of the scheme at $n = 3$, and $h = \pi/10$.

x	**Approximate solution**	**Exact solution**	**Errors**
$\pi/10$	0.31830	0.30902	0.0092806
$\pi/5$	0.60544	0.58779	0.017653
$3\pi/10$	0.83331	0.80902	0.024297
$2\pi/5$	0.97962	0.95106	0.028563

The comparison between the results of the numerical method and the exact solution is presented in Figure. 3.1 at $= 3$, and $h = \pi/10$.

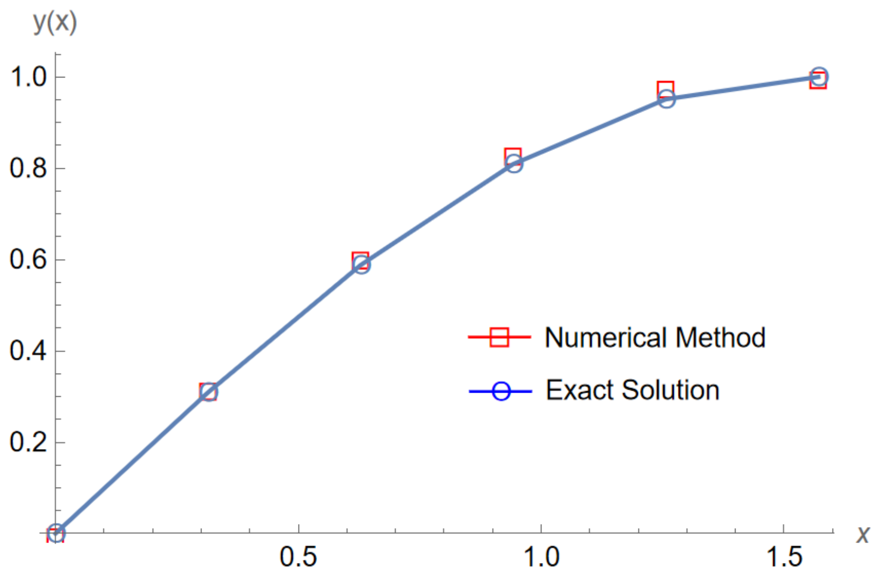

Fig. 3.1. Comparison between the results of the numerical scheme, at $n = 3$, and $h = \pi/10$.

Figure. 3.2, the absolute errors of the numerical method are illustrated at $= 3$, and $h = \pi/10$.

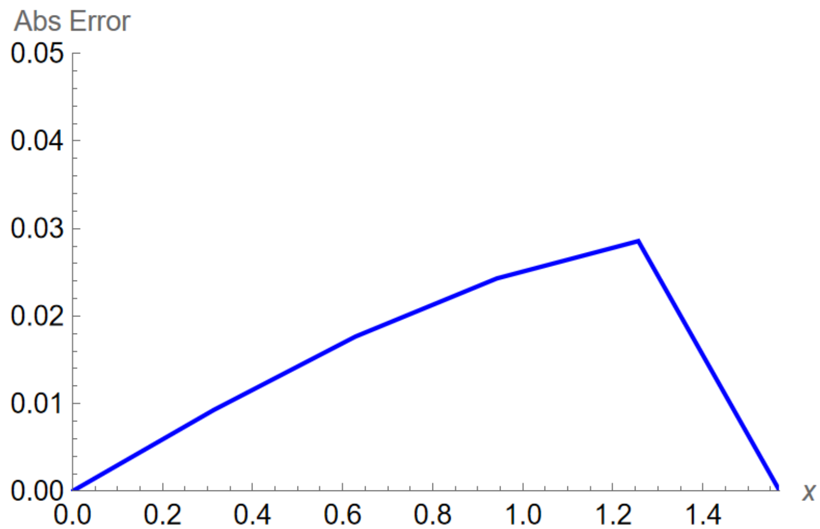

Fig. 3.2. The absolute errors of the scheme at $n = 3$, and $h = \pi/10$.

Example 2. Consider the following Fredholm integral of the second kind equation [10]

$$y(x) = \sqrt{x} - \int_{0}^{1} \sqrt{xt}\, y(t)dt\,, \qquad x \in [0,1] \tag{7}$$

The exact solution of Eq. (7) is $y(x) = \frac{2}{3}\sqrt{x}$.

The computational solutions and the absolute errors are presented in Table 3.2 at $= 3$, and $h = 1/5$.

Table 3. 2. Absolute errors of the scheme at $n = 3$, and $h = 1/5$.

x	Approximate solution	Exact solution	Errors
1/5	0.29814	0.29814	5.55112*E-17*
2/5	0.42164	0.42164	

			0.00000
$3/5$	0.51640	0.51640	1.11022*E-16*
$4/5$	0.59629	0.59629	0.00000

Fig. 3.3 compares the computational Steffensen's results with the exact solution at $= 3$, and $h = 1/5$.

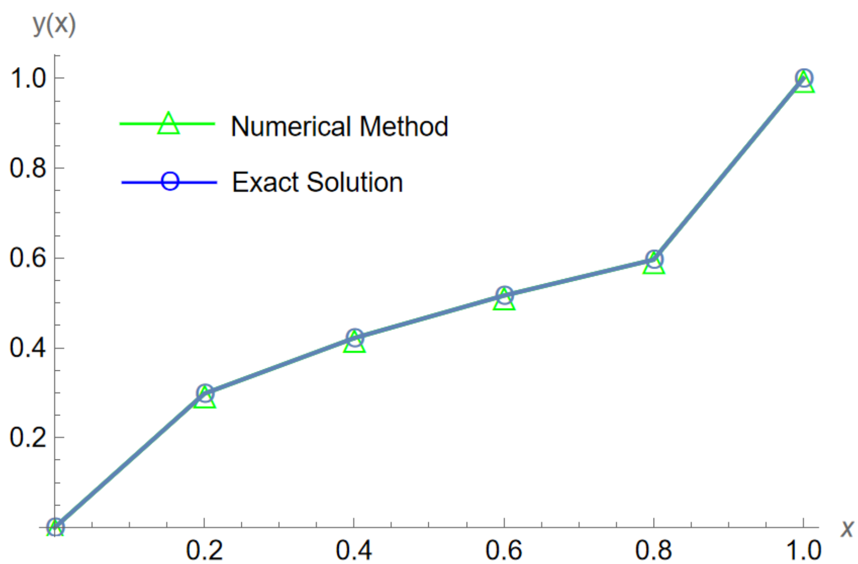

Fig. 3.3. Comparison between the results of the numerical scheme, at $n = 3$, and $h = 1/5$.

Figure. 3.4 illustrates the absolute errors of the numerical method at $= 3$, and $h = 1/5$.

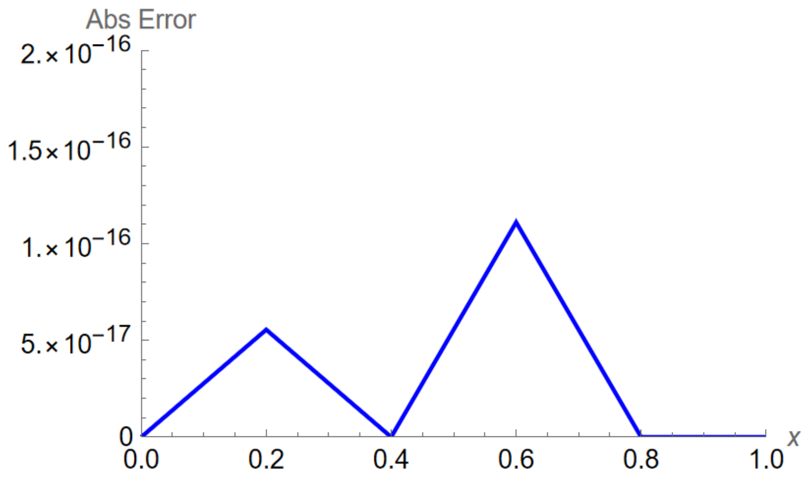

Fig. 3.4. The absolute errors of the scheme at $n = 3$, and $h = 1/5$.

We can see from the above figures that the numerical solutions of Steffensen's method are close to the exact solutions.

5. Conclusion

In this article, a computational Steffensen's method based on open quadrature rule was introduced to solve Fredholm integral equation. The integral term of the Fredholm equation is replaced by an n-point quadrature rule. The presented scheme was demonstrated by numerical examples. The comparison between the approximate solutions and the exact solution was made and showed a good agreement. This proved the feasibility and applicability of this scheme.

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