الطرق العددية للمعادلات التفاضلية الكسرية

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The Numerical Methods of Fractional Differential Equations

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الملخص

لقد أثبتت المعادلات التفاضلية ذات المشتقات من الرتب الغير صحيحة أنها نماذج مناسبة لمجموعة متنوعة من الأحداث الفيزيائية في العديد من المجالات بما في ذلك عمليات الانتشار وقوانين التخميد وميكانيكا الموائع والشبكات العصبية. سأناقش في هذه الدراسة طريقتين عدديتين، طريقة ديثيلم وطريقة آدامز–باشفورث–مولتون لحل المعادلات التفاضلية العادية الكسرية (ODEs) ذات الشروط الإبتدائية.

الكلمات الدالة: طريقة آدامز–باشفورث–مولتون,المشتقة الكسرية كابيتو , طريقة ديثيلم,المعادلات التفاضلية العادية الكسري, المشتق الكسري ليومان لوفيل.

Abstract

Differential equations with non–integer order derivatives have demonstrated are suitable models for a variety of physical events in several fields including diffusion processes and damping laws, fluid mechanics neural networks. In this study, i will discuss two numerical methods Diethelm's method and Adams-Bashforth-Moulton method for solving fractional ordinary differential equations (ODEs) with initial conditions.

Keywords: Adams-Bashforth-Moulton method Caputo fractional derivative Diethelm Method, Fractional ordinary differential equations, Riemann-Liouville fractional derivative.

1. Introduction.

FDEs have been a subject of interest not only among mathematicians but also among physicists and engineers. In fact, we can find numerous applications in economic system

[1], fluid mechanics [2], biology [3], signal processing, dynamics of earthquakes, optics, electromagnetic waves, chaotic dynamics, statistical physics, thermodynamics, neural networks, and so on [4].

Actually, in many cases, it is difficult to obtain the analytical solution. Therefore, the numerical methods are essential for approximation solution of many FDEs. Many approximations have sprung up recently,

There are a number of numerical and analytical methods developed for various types of FDEs, for example, variational iterative method, fractional differential transform method, a domain decomposition method, homotopy perturbation method and power series method [4], the local meshless method based on Laplace transform [5], finite difference methods [6] and finite element methods [7–9].

Hadamard finite–part integral is used by Diethelm [13] to obtain an approximate algorithm for solving fractional differential equation. Podlubny [11] used the Grünwald–Letnikov method to solve FDEs. And very recently Diethelm, Ford and Freed [19] introduced a fractional Adams–type predictor-corrector method for solving FDEs. Lubich [20] wrote the fractional differential equation in the form of an Abel–Volterra integral equation and used the convolution quadrature method to approximate the fractional integral and obtained an approximate solution for fractional differential equations.

In this paper ,I am aiming to use Diethelm's method and Adams–Bashforth–Moulton method to find the general solution of fractional differential equations with initial conditions.

$$\,_0^C D_t^\alpha y(t) = \ f(t, y(t)), \qquad 0 \le t \le T \tag{1.1}$$

$$y^m(0) = y_0{}^m, m = 0,1,2, \ldots, \lceil \alpha \rceil - 1, \tag{1.2}$$

Where $y_0{}^m$ might be random actual numbers, and $\alpha > 0$. here ${}_0^C D_t^\alpha$, represents the differential operator in the Caputo sense, $n - 1 < \alpha < n$.

$$ {}_0^C D_t^\alpha y(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - \tau)^{n-\alpha-1} y^n(\tau) d\tau, $$

where $n = \lceil \alpha \rceil$ is the smallest integer $\geq \alpha$

The following parts are organized as follows. In Section 2 , i introduce the basic definition of fractional caculus. In Section 3, i consider Diethelm's Backward Difference Method. In Section 4, i consider Adams–Bashforth–Moulton method. In Section 5, i will outline the summary of the paper.

2 . Basic definitions

In this section, I set up notations, basic definitions and main properties of Riemann Liouville Integral and derivative and the definition of Caputo fractional derivative is also given.

2.1. Definition ([16] pp.33)

The Riemann–Liouville fractional integral of order $0 < \alpha < 1$, is denoted by the expression:

$$ {}_a^R D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} f(\tau) d\tau . \tag{2.1} $$

2.2 .Definition ([16, pp.35])

Let $\alpha > 0$, the Riemann–Liouville fractional derivative is defined with $n - 1 < \alpha \leq n$ by,

$$ {}_a^R D_t^\alpha f(t) = D^n \left[D_t^{\alpha-n} f(t) \right] = D^n \frac{1}{\Gamma(n-\alpha)} \int_a^t (t - \tau)^{n-\alpha-1} f(\tau) d\tau, \tag{2.2} $$

where $D^n = \frac{d^n}{dt^n}$ denotes the standard nth derivative.

Definition 2.3 [16]

The Caputo fractional derivative of order $\alpha > 0$ is takes the form:

$${}_{a}^{C}D_{t}^{\alpha}f(t)=$$

$$\begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-\tau)^{n-\alpha-1}[D^{n}f(\tau)]d\tau, & \text{where } n-1 < \alpha < n, \\ \frac{d^{n}}{dt^{n}}f(t), & \text{where } \alpha = n. \end{cases} \quad (2.3)$$

The relationship between the Caputo derivative and the Riemann–Liouville derivative is the following, K. Diethelm [12],

Definition 2.4 **[11]**

The Gamma function $\Gamma(x)$ is defined by the integral

$$\Gamma(x) = \int_{0}^{\infty} e^{-t}t^{x-1}\, dt$$

The Gamma function has one of the basic properties:

$$\Gamma(x+1) = x\Gamma(x) \qquad (2.4)$$

Through the application of a partial integration, for the arbitrary $x > 0$, we can manipulate the integral in the definition of the Gamma function which yields:

$$\Gamma(x+1) = \int_{0}^{\infty} e^{-t}t^{x-1}\, dt = [-e^{-t}t^{x-1}]_{0}^{\infty} + x\int_{0}^{\infty} e^{-t}\ t^{x-1}\, dt = x\Gamma(x)$$

Obviously, $\Gamma(1) = 0! = 1$, and using (2.4) we obtain for $x = 1,2,3,\dots$:

$$\Gamma(2) = 1\cdot\Gamma(1) = 1$$
$$\Gamma(3) = 2\cdot\ \Gamma(2) = 2\cdot 1 = 2!$$
$$\Gamma(4) = 3\cdot\Gamma(3) = 3!$$
$$.$$
$$.$$
$$.$$
$$\Gamma(n+1) = \ n\,\Gamma(n)\ = n\cdot(n-1)! = n!$$

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3 Diethelm's Backward Difference Method

The entire procedure is known as the Diethelm fractional backward difference approach because it was first detailed in Diethelm's publication [10], published in 1997.

This section examines the Caputo type fractional ordinary differential equation.

$$_0^C D_t^\alpha y(t) = \lambda y(t) + f(t), \quad 0 < \alpha < 1, \quad \lambda \le 0, \quad 0 \le t \le 1 \qquad (3.1)$$

$$\text{with initial condition: } y(0) = y_0. \qquad (3.2)$$

where α is the order of the derivative, t is the independent variable f is a given function on the interval $[0,1]$, y_0 is the given initial condition, $\lambda \le 0$ and y is the unknown function. From the definition of Riemann–Liouville fractional derivative. This equation can be transformed into a fractional differential equation with the Riemann–Liouville derivative (see Definition (2.1–2.2)),

$$_0^R D_t^\alpha [y - y_0](t) = \lambda y(t) + f(t), \qquad 0 \le t \le 1, \qquad (3.3)$$

$$\text{With the initial condition: } y(0) = y_0. \qquad (3.4)$$

Note that

$$_0^R D_t^\alpha (y_0) = \frac{d}{dt} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} y_0 d\tau$$

$$= \frac{y_0}{\Gamma(1-\alpha)} \frac{d}{dt} \left(\frac{1}{1-\alpha} t^{1-\alpha} \right)$$

$$= \frac{y_0}{\Gamma(1-\alpha)} t^{-\alpha}$$

In order to define a backward difference formula generalization, Diethlm [10] demonstrated a numerical approach that appears to utilize Riemann–Liouville fractional derivative discussed above. The answer's existence and originality have been demonstrated in [15].

Riemann–Liouville fractional derivative is obtained by switching the directions of differentiation and integration, (see definition 2.1) we get,

$$_0^R D_t^\alpha y(t) = \frac{1}{\Gamma(-\alpha)} \int_0^t (t-\tau)^{-\alpha-1} y(\tau) d\tau,$$

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Where the Hadamard finite–part formulation of the integral is used [14].

Let $0 = t_0 < t_1 < \cdots < t_n = 1$ be a partition of [0, 1].

Applying the approximation to the equispaced grid $t_j = j/n$, $j = 1,2,\ldots,n$, $\Delta t = 1/n$, is the time step. We obtain,

$$\,^R_0D_t^\alpha y(t_j) = \frac{1}{\Gamma(-\alpha)} \int_0^{t_j} \frac{y(\tau)}{(t_j-\tau)^{\alpha+1}} d\tau,$$

setting $t_j - \tau = t_j\omega$, we get

$$\,^R_0D_t^\alpha y(t_j) = \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} \int_0^1 \frac{y(t_j-t_j\omega)-y(0)}{\omega^{\alpha+1}} d\omega = \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} \int_0^1 g(\omega)\omega^{-\alpha-1} d\omega,$$

where

$$g(\omega) = y(t_j - t_j\omega) - y(0).$$

In this section, a compound quadrature formula is used in place of the integral. [13], and equally spaced nodes $0, \frac{1}{j}, \frac{2}{j}, \ldots, 1$ for each j, gives

$$\,^R_0D_t^\alpha y(t_j) = \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} \left[\sum_{k=0}^{j} \alpha_{kj} y(t_j - t_k) + R_j(g) \right],$$

Thus the approximation can be represented by a quadrature formula of a product trapezoidal form

$$Q_j[g] := \sum_{k=0}^{j} \alpha_{kj} g\left(\frac{k}{j}\right) \approx \int_0^1 g(\omega)\omega^{-\alpha-1} d\omega,$$

Where

$$\int_0^1 g(\omega)\tau^{-\alpha-1} d\tau = Q_j[g] + R_j(g),$$

And the remainder term $R_j(g)$ satisfies

$$\|R_j(g)\| \le Cj^{\alpha-2} \sup_{0 \le t \le T} \|y''(t_j - t_j\omega)\|.$$

Thus,

$$\,^R_0D_t^\alpha y(t_j) =$$

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$$\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=0}^{j}(-\alpha)(1-\alpha)j^{-\alpha}\,\alpha_{kj}\,y(t_j-t_k)+\frac{t_j^{-\alpha}}{\Gamma(-\alpha)}\,R_j(g)$$

$$=\Delta t^{-\alpha}\sum_{k=0}^{j}\frac{(-\alpha)(1-\alpha)j^{-\alpha}\,\alpha_{kj}}{\Gamma(2-\alpha)}y(t_j-t_k)$$

$$+\frac{t_j^{-\alpha}}{\Gamma(-\alpha)}\,R_j(g)$$

$$=\Delta t^{-\alpha}\sum_{k=0}^{j}\omega_{kj}\,y(t_j-t_k)+\frac{t_j^{-\alpha}}{\Gamma(-\alpha)}\,R_j(g).$$

Here

$$\Gamma(2-\alpha)\omega_{kj}=-\alpha(1-\alpha)j^{-\alpha}\alpha_{kj}\,, \tag{3.5}$$

Where the weights ω_{kj} satisfies that [13]

$$\Gamma(2-\alpha)\omega_{kj}=$$

$$\begin{cases} 1, & \text{for } k=0,\\ -2k^{1-\alpha}+(k-1)^{1-\alpha}+(k+1)^{1-\alpha}, & \text{for } k=1,2,\dots,j-1, \\ -(\alpha-1)k^{-\alpha}+(k-1)^{1-\alpha}-k^{1-\alpha}, & \text{for } k=j, \end{cases} \tag{3.6}$$

and α_{kj} satisfies

$$\alpha(1-\alpha)j^{-\alpha}\alpha_{kj}=$$

$$\begin{cases} -1, & \text{for } k=0,\\ 2k^{1-\alpha}-(k-1)^{1-\alpha}-(k+1)^{1-\alpha}, & \text{for } k=1,2,\dots,j-1, \\ (\alpha-1)k^{-\alpha}-(k-1)^{1-\alpha}+k^{1-\alpha}, & \text{for } k=j. \end{cases} \tag{3.7}$$

Now this section considers the finite difference method of

$$_0^R D_t^\alpha[y-y_0](t)=\lambda y(t)+f(t),\quad\text{at}\quad t=t_j,\text{get}$$

$$_0^R D_t^\alpha[y(t)-y(0)]|_{t=t_j}=\lambda y(t_j)+f(t_j),$$

$$\Delta t^{-\alpha}\sum_{k=0}^{j}\omega_{kj}\big[\,y(t_j-t_k)-y(0)\big]+\frac{t_j^{-\alpha}}{\Gamma(-\alpha)}\,R_j(g)$$

$$=\lambda y(t_j)+f(t_j),$$

or

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$$\left.{}_0^R D_t^\alpha [y(t) - y(0)]\right|_{t=t_j} = \Delta t^{-\alpha} \sum_{k=0}^{j} \omega_{kj} [y(t_j - t_k) - y(0)] + \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} R_j(g),$$

Denote $y_j \approx y(t_j)$ as the approximation of $y(t_j)$, this study can clarify

$$\Delta t^{-\alpha} \sum_{k=0}^{j} \omega_{kj} [y_{j-k} - y_0] = \lambda y_j + f_j,$$

Let k=0, we get

$$(\omega_{0j} - \Delta t^\alpha \lambda) y_j = \Delta t^\alpha f_j - \sum_{k=1}^{j} \omega_{kj} y_{j-k} + \sum_{k=0}^{j} \omega_{kj} y_0.$$

From (3.5) we can find

$$\sum_{k=0}^{j} \omega_{kj} = \frac{-\alpha(1-\alpha)j^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{j} \alpha_{kj}.$$

But

$$\sum_{k=0}^{j} \alpha_{kj} = \int_0^1 u^{-\alpha-1} du = -\frac{1}{\alpha}.$$

Thus

$$\sum_{k=0}^{j} \omega_{kj} = \frac{-\alpha(1-\alpha)j^{-\alpha}}{\Gamma(2-\alpha)} \left(-\frac{1}{\alpha}\right) = \frac{j^{-\alpha}}{\Gamma(1-\alpha)}.$$

Diethelm's numerical approach for the equations (3.3) and (3.4) is provided by the implicit formula below:

$$y_j = (\omega_{0j} - \Delta t^\alpha \lambda)^{-1} [\Delta t^\alpha f_j - \sum_{k=1}^{j} \omega_{kj} y_{j-k} +$$
$$+ \frac{j^{-\alpha}}{\Gamma(1-\alpha)} y_0]. \tag{3.8}$$

4. Adams–ashforth–Moulton method [12]

This section will present the algorithm for the fractional differential equation in the Caputo type:

$$_0^C D_t^\alpha y(t) = f(t, y(t)), \tag{4.1}$$

with the initial condition

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$$D^k y(0) = b_k \ , \quad k = 0,1,2,\dots,n-1 \ , \tag{4.2}$$

The algorithm to solve the fractional differential equation of Caputo type is based on the fractional formulation of the classical Adams–Bashforth–Moulton method. In particular by using the formulation of the problem in Abel–Volterra integral form, i.e

$$y(t) = \sum_{k=0}^{n-1} \frac{D^k y(0) t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau \tag{4.3}$$

In order to discuss differential equations of fractional order it is necessary to review the classical differential equations and the methods used to numerically solve these equations. From the classical algorithms it is possible to extend the resulting formulas to the fractional differential equation so it is important to understand this common background for numerical methods. It must be noted that many classical numerical schemes can be extended in more than one way which can led to issues within literature as different equations could be conveyed in a similar manner creating a potential source of confusion. For example, the fractional Adams–Moulton rules of Galeone and Garrappa [17] do not coincide with the methods of the same name as it will be demonstrated later in this section below.

4.1. Classical Formulation

Diethelm [12] identifies that the classical Adams–Bashforth–Moulton algorithm for first-order equations should be reviewed to enable a starting point by using the familiar initial-value problem for the first-order differential equation

$$Dy(t) = f(t, Y(t)), \tag{4.4}$$

$$y(0) = y_0. \tag{4.5}$$

It is assumed that the function f will be a unique solution that exists on some interval $[0, T]$. Following Hairer & Wanner [18], Diethelm [12] advises to use the predictor–corrector technique of Adams where it is assumed that for simplicity that mathematician is working on a uniform grid $\{t_i = ih : i = 0, 1, \ldots, N\}$ with some integer N and $h = T/N$. While in some applications it would be more efficient to utilise a non–uniform grid and this will be identified to the reader and thus a generalised sense of numerical approximation formulas will be utilised. When reviewing the properties of the scheme the author will restrict themselves to the isometric case.

Basically it assumed that the approximations have already been calculated as $y_{i \approx} y(t_i)$, $(i = 1, 2, \ldots, n)$. While trying to obtain the approximation y_{n+1} by means of the equation

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(z, y(z)) dz. \qquad (4.6)$$

Following the integration of equation (4.4) on the interval $[t_n, t_{n+1}]$ without knowing either of the expressions on the right–hand side of equation (4.6) exactly. Yet there is an approximation for $y(t_n)$, namely y_n that can exploit instead. The integral is then replaced by the two–point trapezoidal quadrature formula

$$\int_a^b g(z) dz \approx \frac{b-a}{2} (g(a) + g(b)), \qquad (4.7)$$

Thus giving an equation for the unknown approximation y_{k+1}, it being

$$y_{n+1} = y_n + \frac{t_{n+1} - t_n}{2} \left(f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1})) \right). \qquad (4.8)$$

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Again $y(t_n)$ and $y(t_{n+1})$ are replaced by their approximations y_n and y_{n+1} respectively and this produces the equation for the implicit one–step Adams–Moulton method, which is

$$y_{n+1} = y_n + \frac{t_{n+1}-t_n}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right).$$

Diethelm [12] advises that the so–called predictor or preliminary approximation y_{n+1}^p is similarly obtained by only replacing the trapezoidal quadrature formula in the rectangle rule giving the explicit forward Euler method to produce the following formula:

$$y_{n+1}^p = y_n + hf(t_n, y_n). \tag{4.9}$$

And

$$y_{n+1} = y_n + \frac{h}{2}\left(f(t_n, y_n) + f\left(y_{n+1}, y_{n+1}^p\right)\right), \tag{4.10}$$

This approach is known as the one–step Adams–Bashforth–Moulton method,

The convergence order of (4.10) is 2, i.e,

$$\max_{i=1,2,\dots,N}|y(t_i) - y_i| = O(h^2). \tag{4.11}$$

Where $y(t_i)$ is an exact solution and y_i is an approximate solution.

4.2. Fractional Formulation

From the classical algorithms it is possible to transfer the essential concepts over to the fractional–order problems of courses with some necessary adaptions. The key to addressing this application to fractional–order problems is to develop an equation which is similar to (4.10) according Diethelm [12] but the equation will be different due to the range of integration which now starts at 0 instead of t_k.

By using the product trapezoidal quadrature formula to replace the integral, for example the nodes $t_i\{ i = 0,1,2, \dots, k+1 \}$ and thus interpret the function $(t_{k+1} - \cdot)$ as a weight function for the integral. By apply the approximation

$$\int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} g(t) dt \approx \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} g_{k+1}^*(t) dt,$$

where g_{k+1}^* is the piecewise linear interpolant for g with nodes and knots chosen at the $t_i, i = 0,1,2,\dots,k+1$. From this construction it demonstrates that the weighted trapezoidal quadrature formula can be represented as a weighted sum of function values of the integrand g, taken at the points t_i. Explicitly, the integral on the right–hand side of (4.13) can be expressed as

$$\int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} g_{k+1}^*(t) dt = \sum_{i=0}^{k+1} a_{i,k+1} \, g(t_i). \qquad (4.12)$$

Where

$$a_{i,k+1} = \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} \phi_{i,k+1}(t) dt, \qquad (4.13)$$

and

$$\phi_{i,k+1}(t) = \begin{cases} \dfrac{t - t_{i-1}}{t_i - t_{i-1}}, & \text{if } t_{i-1} < t \le t_i, \\ \dfrac{t_{i+1} - t}{t_{i+1} - t_i}, & \text{if } t_i < t < t_{i+1}, \\ 0, & \text{else}. \end{cases} \qquad (4.14)$$

This is clear because the functions $\phi_{i,k+1}$ satisfy

$$\phi_{i,k+1}(t_\mu) = \begin{cases} 1 & \text{if} & i \ne \mu \\ 0 & \text{if} & i = \mu \end{cases}$$

And that they are continuous and piecewise linear with breakpoints at the nodes t_μ, and thus must integrated exactly by the developed formula.

An easy explicit calculation produces that, for an arbitrary choice of the t_i, (4.13) and (4.14) result in (4.15)

$$a_{0,k+1} = \frac{(t_{k+1} - t_1)^{\alpha+1} + t_{k+1}^{\alpha}[t_1 + t_1 - t_{k+1}]}{t_1 \alpha(\alpha + 1)},$$

$$a_{i,k+1} = \frac{(t_{k+1} - t_{i-1})^{\alpha+1} + (t_{k+1} - t_i)^{\alpha}[\alpha(t_{i-1} - t_i) + t_{i-1} - t_{k+1}]}{(t_i - t_{i-1})\alpha(\alpha + 1)} +$$

$$\frac{(t_{k+1} - t_{i+1})^{\alpha+1} - (t_{k+1} - t_i)^{\alpha}[\alpha(t_i - t_{i+1}) - t_{i+1} + t_{k+1})}{(t_{i+1} - t_i)\alpha(\alpha + 1)}, \qquad 1 \le i \le k,$$

$$a_{k+1,k+1} = \frac{(t_{k+1} - t_k)^{\alpha}}{\alpha(\alpha+1)} \qquad\qquad\qquad (4.15)$$

The isometric nodes ($t_i = ih$ with some fixed h) are reduced to the following equations. This then provides a factional variant of the one–step Adams–Moulton method by providing the correct formula which is

$$a_{i,k+1} = \begin{cases} \frac{h^{\alpha}}{\alpha(\alpha+1)}(k^{\alpha+1} - (k - \alpha)(k + 1)^{\alpha}) & \text{if } i = 0 \\ \frac{h^{\alpha}}{\alpha(\alpha+1)}((k - i + 2)^{\alpha+1} + (k - i)^{\alpha+1} \\ -2(k - i + 1)^{\alpha+1}) & \text{if } 1 \le i \le k \\ \frac{h^{\alpha}}{\alpha(\alpha+1)} & \text{if } i = k + 1 \end{cases} \qquad (4.16)$$

This then provides a factional variant of the one –step *Adams–Moulton method* by providing the correct formula which is

$$y_{k+1} = \sum_{i=0}^{m-1} \frac{t_{k+1}^i}{i!} y_0^{(i)} + \frac{1}{\Gamma(n)} \left(\sum_{i=0}^{k} a_{i,k+1} f(t_i, y_i) + a_{k+1,k+1} f(t_{k+1}, y_{k+1}^p) \right). \quad (4.17)$$

What remains are the resolution of the predictor formula and thus the required calculation of the value y_{k+1}^p. The same concept that was utilized to generalize the Adams–Moulton technique is applied to the one–step Adams–Bashforth method by replacing the integral with the product of rectangle rule

$$\int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} g(t)dt \approx \sum_{i=0}^{k} b_{i,k+1} g(t_i), \qquad\qquad (4.18) \text{ where}$$

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$$b_{i,k+1} = \int_{t_i}^{t_{i+1}} (t_{k+1} - t)^{\alpha-1} dt = \frac{(t_{k+1}-t_i)^{\alpha} - (t_{k+1}-t_{i+1})^{\alpha}}{\alpha} . \qquad (4.19)$$

Similarly to the method utilized in the equations (4.16)–(4.18) the weight can be consequential calculated. Yet there is a requirement to utilize a piecewise constant approximation and not a piecewise linear one, and hence there is a requirement to

$$b_{i,k+1} = \frac{h^{\alpha}}{\alpha}((k+1-i)^{\alpha} - (k-i)^{\alpha}). \qquad (4.20)$$

Thus, the predictor y_{k+1}^{p} is determined by the fractional Adams–Bashforth method

$$y_{k+1}^{p} = \sum_{i=0}^{m-1} \frac{t_{k+1}^{i}}{i!} y_0^{(i)} + \frac{1}{\Gamma(n)} \sum_{i=0}^{k} b_{i,k+1} f(t_i, y_i) , \qquad (4.21)$$

The fractional Adams–Bashforth–Moulton method, is therefore completed and described by the formula expressions (4.21) and (4.17) with the weights $a_{i,k+1}$ and $b_{i,k+1}$ as defined according to (4.15) and (4.20), respectively.

5 .Conclusion

This paper has presented the analytical and numerical scheme for the solution of the fractional ordinary differential equations by using two numerical methods , Diethelm's method and Adams–Bashforth–Moulton method , as numerical methods for approximate solutions accordingly of type Liouville–Caputo the methods that are more accurate and cost effective in mathematical modelling. Diethelm provides that the error behaves as $O(h^{2-\alpha})$ when using functions that are sufficiently smooth. The method is analysed for $0 < \alpha < 1$. Diethelm provides that the extension to $1 < \alpha < 2$ should not present major difficulty.

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