المؤتمر العلمي الدولي الثالث للهندسة و العلوم

**The 3RD Scientific International Conference in Engineering & Science**

**http://bwu.edu.ly/icse2024**
Received 25/07/2024 Revised 23/08/2024 Published 10/09/2024
**icse@bwu.edu.ly**

# New Results on Investigated Integral Representation and Convolution Characterization and Differential Subordination for Univalent Functions

**Eman Khleifa Shmella [1] Aisha Ahmed Amer[2]**

[1]Mathematics Dept-Faculty of Science, Zliten - Alasmarya Islamic University,Libya

[2]Mathematics Dept-Faculty of Sciences , Khoms- El Mergib University ,Libya

[1]aymankhlyfh272@gmail.com [2] aisha.amer@elmergib.edu.ly

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**Abstract:** In this work, we are investigated integral representation and Convolution characterization and Results of Differential Subordination for functions belong to $\mathcal{R}_\theta(\psi)$by introduce generalized derivative operator ,where $\mathcal{R}_\theta(\psi)$ denote to the class of all analytic normalized functions in $\mathbb{U}$.

**Keywords**: differential subordination, Integral Representation, normalized functions.

## 1. Introduction

Let $f$ denote of the analytic functions in the class $\mathcal{A}$ [1] of the form :

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathbb{U}) \ , \quad (1)$$

in the open unit disk $\mathbb{U} = \{z: |z| < 1; z \in \mathbb{C}\}$ ,where $a_k$ is a complex number .and denote $\mathcal{S}$ the subclass of $\mathcal{A}$ in $\mathbb{U}$ where $\mathcal{S}$ consisting of univalent functions and let the familiar subclass $C$ of $\mathcal{S}$ whose members are convex functions in $\mathbb{U}$ .

Now, let $\mathcal{M}$denote the class of analytic functions $\psi(z)$ in $\mathbb{U}$, normalized by $|\psi(z)| \le 1$ and $\psi(0) = 1$. all univalent functions $\psi$ belong to the subclass $\mathcal{N}$ of $\mathcal{M}$ for which $\psi(\mathbb{U})$ is a convex domain .

Now , denote by $\mathcal{P}$ the well –known class of analytic functions $p(z)$ $\forall z \in \mathbb{U}$ with

$$Re\big(p(z)\big) > 0 \quad and \ \ p(0) = 1.$$

And denote by $\mathcal{B}$ the class of analytic functions $\omega(z)$ in $\mathbb{U}$ $\forall z \in \mathbb{U}$ with [1] $|\omega(z)| < 1 \quad and \ \omega(0) = 0$ .

Recently, Silverman and Silvia [2]considered the following class of functions:

$$\mathcal{L}_\theta = \left\{ f: f \in \mathcal{A} \ and \ \ Re\left( f'(z) + \frac{1+e^{i\theta}}{2} z f''(z) \right) > 0 \right\},$$

where $\theta \in (-\pi, \pi]$ . If $b \to \infty$ .for this class of functions ,they obtained extreme points and convolution characterizations .[3],on the other hand ,studied the function class $\mathcal{L}\mathcal{p}_\theta$ given by

$$\mathcal{L}\mathcal{p}_\theta=\left\{f: f\in\mathcal{A}\ and\ f'(z)+\frac{1+e^{i\theta}}{2}z f''(z)\prec Q(z)\right\},$$

where $\theta\in(-\pi,\pi]$. The function $Q(z)$ $\forall z\in\mathbb{U}$ where $Q(0)=1$ *and*

$$Q(z)=1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\rightarrow(2)$$

maps onto the domain given by

$\Omega=\{w: w\in\mathbb{C}\ \ and\ \ |w-1|<Re\,(w)\}$.

Now ,if the function $f$ *and* $g$ are analytic in $\mathbb{U}$ ,then we say $f$ is subordinate to $g$ in $\mathbb{U}$,written as $f\prec g$ if there is a Schwarz function $v(z)$ analytic in $\mathbb{U}$, with $|v(z)|<1$, so that $f(z)=g(v(z))$ ; $z\in\mathbb{U}$.

Furthermore ,If the function $g$ is univalent in $\mathbb{U}$ then the subordination $f(z)\prec g(z)$ is equivalent to

$f(0)=g(0)$ *and* $f(\mathbb{U})=g(\mathbb{U})$ [4].

The Hadamard product of two analytic functions $f$and $g$ denoted by $f*g$ , where $f(z)$ of the form (1) *and*

$g(z)=z+\sum_{k=2}^{\infty}b_kz^k$; $(z\in\mathbb{U})$,is defined by

$$(f*g)(z)=f(z)*g(z)=z+\sum_{k=2}^{\infty}a_kb_kz^k.$$

In light of product, Amer and Darus [5] they have recently introduced a new generalized derivative operator.

**Definition 1:**

For $f\in\mathcal{A}$ the operator $I^m(\lambda_1,\lambda_2,\ell,n)$ is defined by $I^m(\lambda_1,\lambda_2,\ell,n):\mathcal{A}\longrightarrow\mathcal{A}$.

$$I^m(\lambda_1,\lambda_2,\ell,n)f(z)=\varphi^m(\lambda_1,\lambda_2,\ell)(z)*R^nf(z),$$

where $\lambda_2\geq\lambda_1\geq0,\ \ell\geq0$ and $m\in N_0=\{0,1,2,\ldots\}$ and $R^nf(z)$ denotes the Rusehewey h derivative operator all $z\in\mathbb{U}$ and given by

$$R^nf(z)=z+\sum_{k=2}^{\infty}c(n,k)\ \ a_kb_kz^k,$$

where $c(n,k)=\frac{(n+1)_{k-1}}{(1)_{k-1}}$ *and* $n\in N_0$ .

If $f(z)$ given by (1), then we easily find from

$$I^m(\lambda_1,\lambda_2,\ell,n)f(z)=\varphi^m(\lambda_1,\lambda_2,\ell)(z)*R^nf(z),$$

That

$$I^m(\lambda_1,\lambda_2,\ell,n)f(z)=z+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+\ell)^{m-1}}{(1+\ell)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)a_kz^k$$

,where

$n,m\in N_0=\{0,1,2,\ldots\}$ *and* $\lambda_2\geq\lambda_1\geq0,\ \ell\geq0.$

Using simple computation one obtains the next result

$$(\ell+1)I^{m+1}(\lambda_1,\lambda_2,\ell,n)f(z)$$

$$=(1+\ell-\lambda_1)(I^m(\lambda_1,\lambda_2,\ell,n)*\varphi^1(\lambda_1,\lambda_2,\ell)(z))f(z)$$

$$+\lambda_1z\left(I^m(\lambda_1,\lambda_2,\ell,n)*\varphi^1(\lambda_1,\lambda_2,\ell)f(z)\right)',\rightarrow(3)$$

where $\varphi^1(\lambda_1,\lambda_2,\ell)(z)$ analytic function given by

$$\varphi^1(\lambda_1,\lambda_2,\ell)(z) = z + \sum_{k=2}^{\infty} \frac{1}{(1+\lambda_2(k-1))} z^k. \to (4)$$

Now, from equation (2) and (4),we have

$$\left(I^m(\lambda_1,\lambda_2,\ell,n) * \varphi^1(\lambda_1,\lambda_2,\ell)f(z)\right)' =$$

$$\left(\left(z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+\ell)^{m-1}}{(1+\ell)^{m-1}(1+\lambda_2(k-1))^m} c(n,k) a_k z^k\right) * \left(z + \sum_{k=2}^{\infty} \frac{1}{(1+\lambda_2(k-1))} z^k\right)\right)'$$

$$= \left(z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+\ell)^{m-1}}{(1+\ell)^{m-1}(1+\lambda_2(k-1))^m} c(n,k) a_k z^k\right)'$$

$$= \left(I^m(\lambda_1,\lambda_2,\ell,n)f(z)\right)'$$

So, by using equation (3), we obtain

$$z\left(I^m(\lambda_1,\lambda_2,\ell,n)f(z)\right)' =$$

$$\frac{(\ell+1)}{\lambda_1} I^{m+1}(\lambda_1,\lambda_2,\ell,n)f(z) - \frac{(1+\ell-\lambda_1)}{\lambda_1}(I^m(\lambda_1,\lambda_2,\ell,n)f(z). \to (5)$$

**Definition 2: [1]**

Let $\theta \in (-\pi,\pi]$ and let $\psi \in \mathcal{M}$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}_\theta(\psi)$ if the following differential subordination is satisfied:

$$f'(z) + \frac{1+e^{i\theta}}{2} z f''(z) \prec \psi(z), (z \in \mathbb{U}). \quad \to (6)$$

Consider the function:

$$\psi_0(z) = \frac{1+z}{1-z}$$

So the corresponding class $\mathcal{R}_\theta(\psi_0)$ reduce to the class $\mathcal{L}_\theta$ . and the class $\mathcal{R}_\theta(Q)$ reduces to function class $\mathcal{L}\wp_\theta$; the function $Q$ is defined by (2).

We now define the function class $\mathcal{R}$ by

$$\mathcal{R} = \mathcal{R}_0(\psi_0) = \{f : f \in \mathcal{A} \text{ and } Re(f'(z) + z f''(z)) > 0\},$$

was investigated by Chichra [6] and also by Singh and Singh [7]. Another function class $\mathcal{R}_\beta$ given by

$$\mathcal{R}_\beta = \{f : f \in \mathcal{A} \text{ and } Re(f'(z) + z f''(z)) > \beta\}, \to (7)$$

which was considered by Silverman [8], can also be obtained from $\mathcal{R}_\theta(\psi)$ upon setting $\theta = 0$ and $\psi = \psi_\beta$ ;$(0 \le \beta < 1)$where $\psi_\beta = \frac{1+(1-2\beta)z}{1-z}$.

**Lemma 1: [9]**

Let $T$ be a convex function where $T(0) = a$ and $\tau \in \mathbb{C}^*$ with $Re\, \tau \ge 0$. If the function $p(z)\ \forall\ z \in \mathbb{U}$ defined by

$$p(z) = a + p_n z^n + p_{n+1} z^{n+1} + \cdots,$$

is analytic in $\mathbb{U}$ an

$$p(z) + \frac{1}{\tau} z p'(z) \prec T(z),$$

$$\therefore p(z) \prec q(z) \prec T(z), \text{ where}$$

$$q(z) = \frac{\tau}{n\, z^{\tau/n}} \int_0^z T(\zeta)\zeta^{\tau/n - 1}\, d\tau.$$

**2. Convolution Characterization, Integral Representation and Results Involving Differential Subordination:**

**Theorem 1**

If $\psi \in \mathcal{M}$. A sufficient and necessary condition for a function $f \in \mathcal{A}$ to be in the class $\mathcal{R}_\theta(\psi)$ is given by

$$\frac{1}{z}\left(\left((\ell+1)I^{m+1}(\lambda_1,\lambda_2,\ell,n)f(z) + (\lambda_1 - 1 - \ell)\left(I^m(\lambda_1,\lambda_2,\ell,n)f(z)\right)\right) * \frac{z - z^2 e^{i\theta}}{(1-z)^3}\right) \ne \lambda_1\, \psi(e^{i\alpha}),$$

where $\theta \in (-\pi,\pi]$ , $\alpha \in [0,2\pi)$ and $z \in \mathbb{U}$.

**Proof**

From ( Definition 2) $f \in \mathcal{R}_\theta(\psi)$ if and only if

$$f'(z) + \frac{1+e^{i\theta}}{2} z\, f''(z) \neq \psi(z)$$

$$f'(z) + \frac{1+e^{i\theta}}{2} z f''(z) \neq \psi\left(e^{i\alpha}\right)$$

Since

$$f'(z) + \frac{1+e^{i\theta}}{2} z\, f''(z)$$

$$= f'(z)\, \frac{2+e^{i\theta}-e^{i\theta}}{2} + \frac{1+e^{i\theta}}{2} z\, f''(z)$$

$$= f'(z)\left(\frac{1+e^{i\theta}}{2} + \frac{1-e^{i\theta}}{2}\right) + \frac{1+e^{i\theta}}{2} z\, f''(z)$$

$$= \left(\frac{1+e^{i\theta}}{2}\right)(zf'(z))' + \frac{1-e^{i\theta}}{2} f'(z)) \neq \psi\left(e^{i\alpha}\right)$$

$$\therefore f'(z) + \frac{1+e^{i\theta}}{2} z\, f''(z) =$$

$$\left(f(z)\frac{1-e^{i\theta}}{2}\right)' + \left(\frac{1+e^{i\theta}}{2} zf'(z)\right)' \neq \psi\left(e^{i\alpha}\right) \longrightarrow (8)$$

Now, let

$$zf'(z) = I^m(\lambda_1, \lambda_2, \ell, n) f(z) * \frac{z}{(1-z)^2} \longrightarrow (9)$$

and

$$f(z) = I^m(\lambda_1, \lambda_2, \ell, n) f(z) * \frac{z}{1-z}, \longrightarrow (10)$$

By using (9) and (10) in (8), we get

$$f'(z) + \frac{1+e^{i\theta}}{2} z\, f''(z)$$

$$= \left(I^m(\lambda_1, \lambda_2, \ell, n) f(z) * \frac{1-e^{i\theta}}{2} \frac{z}{1-z}\right)' + \left(I^m(\lambda_1, \lambda_2, \ell, n) f(z) * \frac{1+e^{i\theta}}{2} \frac{z}{(1-z)^2}\right)'$$

$$= I^m(\lambda_1, \lambda_2, \ell, n) f'(z) *$$

$$\left(\frac{1-e^{i\theta}}{2} \frac{z}{1-z} + \frac{1+e^{i\theta}}{2} \frac{z}{(1-z)^2}\right)' \neq \psi\left(e^{i\alpha}\right).$$

That is equivalently,

$$\left(I^m(\lambda_1, \lambda_2, \ell, n) f(z) * \left(\frac{(1-z)\,(1-e^{i\theta})z + (1+e^{i\theta})z}{2(1-z)^2}\right)\right)'$$

$$\neq \psi\left(e^{i\alpha}\right)$$

$$\left(I^m(\lambda_1, \lambda_2, \ell, n) f(z) * \frac{(2z - z^2(1-e^{i\theta})}{2(1-z)^2}\right)' \neq \psi\left(e^{i\alpha}\right)$$

$$\left(I^m(\lambda_1, \lambda_2, \ell, n) f(z) * \frac{z - z^2(\frac{1-e^{i\theta}}{2})}{(1-z)^2}\right)' \neq \psi\left(e^{i\alpha}\right)$$

$$\left(I^m(\lambda_1, \lambda_2, \ell, n) f(z)\right)' *$$

$$\frac{(1-z)^2(1-z+ze^{i\theta}) - (-2)(1-z)\left(z - z^2\left(\frac{1-e^{i\theta}}{2}\right)\right)}{(1-z)^4}$$

$$\neq \psi\left(e^{i\alpha}\right)$$

$$\Rightarrow \left(I^m(\lambda_1, \lambda_2, \ell, n) f(z)\right)' * \frac{(1-z)}{(1-z)^4}$$

$$\left(1 - z + ze^{i\theta} - z + z^2 - z^2e^{i\theta} + 2z - z^2 - z^2e^{i\theta}\right) \neq \psi\left(e^{i\alpha}\right)$$

$$\frac{1}{z}\left(\left(I^m(\lambda_1, \lambda_2, \ell, n) f(z)\right)' * \frac{z - z^2e^{i\theta}}{(1-z)^3}\right) \neq \psi\left(e^{i\alpha}\right)$$

By using (5), we obtain

$$\frac{1}{z}\left(\left((\ell+1) I^{m+1}(\lambda_1, \lambda_2, \ell, n) f(z) + (\lambda_1 - 1 - \ell)\left(I^m(\lambda_1, \lambda_2, \ell, n) f(z)\right)\right) * \frac{z - z^2e^{i\theta}}{(1-z)^3}\right) \neq \lambda_1\, \psi\left(e^{i\alpha}\right).$$

**Corollary 1: [1]**

If $\psi \in \mathcal{M}$. A sufficient and necessary condition for a function $f \in \mathcal{A}$ to be in the class $\mathcal{R}_\theta(\psi)$ is given by

$$\frac{1}{z}\left(I^m(\lambda_1,\lambda_2,\ell,n)f(z) * \frac{z+z^2e^{i\theta}}{(1-z)^3}\right) \neq \psi(e^{i\alpha}),$$

where $\theta \in (-\pi,\pi]$, $\lambda_1,\lambda_2,\ell,n = 0$ and $z \in \mathbb{U}; \alpha \in [0,2\pi)$.

**Theorem 2**

If $\theta \in (-\pi,\pi)$ and let $\psi \in \mathcal{M}$. Suppose also that

$$\tau = \frac{2}{1+e^{i\theta}}.$$

Then $f \in \mathcal{R}_\theta(\psi)$ if and only if there exists $\omega \in \mathcal{B}$ such that the following equality :

$$I^m(\lambda_1,\lambda_2,\ell,n)f(z) = \int_0^z \frac{\tau}{\eta^\tau}\left(\int_0^\eta \zeta^{\tau-1}\psi(\omega(\zeta))d\zeta\right)d\eta; z \in \mathbb{U}.$$

**Proof:**

from (Definition 2) $f \in \mathcal{R}_\theta(\psi) \Leftrightarrow$ there exists $\omega \in \mathcal{B}$ such that

$$f'(z) + \frac{1+e^{i\theta}}{2} z f''(z) = \psi(\omega(z)) \longrightarrow (11)$$

By using (8) in the above equality (11), we obtain

$$\frac{1-e^{i\theta}}{2} f'(z) + \frac{1+e^{i\theta}}{2}(z f'(z))' = \psi(\omega(z))$$

Now, we have a derivative operator $I^m(\lambda_1,\lambda_2,\ell,n)f(z)$;

$$I^m(\lambda_1,\lambda_2,\ell,n)f(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+\ell)^{m-1}}{(1+\ell)^{m-1}(1+\lambda_2(k-1))^m} c(n,k)a_k z^k$$

, where $n,m \in N_0 = \{0,1,2,\ldots\}$ and $\lambda_2 \geq \lambda_1 \geq 0, \ell \geq 0$.

It follows that

$$\frac{2}{1+e^{i\alpha}}\left(\frac{1-e^{i\theta}}{2}\right)\left( (I^m(\lambda_1,\lambda_2,\ell,n)f(z)\right)'$$

$$+\frac{2}{1+e^{i\theta}}\left(\frac{1+e^{i\theta}}{2}\right)(z\ (I^m(\lambda_1,\lambda_2,\ell,n)f(z))')'$$

$$= \frac{2}{1+e^{i\theta}}\ \psi(\omega(z)).$$

$$\Longrightarrow \left(\frac{1-e^{i\theta}}{1+e^{i\theta}}\right)(I^m(\lambda_1,\lambda_2,\ell,n)f(z))'$$

$$+(z\ (I^m(\lambda_1,\lambda_2,\ell,n)f(z))')' = \frac{2}{1+e^{i\theta}}\ \psi(\omega(z)).$$

$$\because\ \tau = \frac{2}{1+e^{i\alpha}}\quad ; \alpha \neq \pi.$$

we obtain

$$\left(\frac{1-e^{i\theta}}{2}\right)\tau\ (I^m(\lambda_1,\lambda_2,\ell,n)f(z))'$$

$$+(z\ (I^m(\lambda_1,\lambda_2,\ell,n)f(z))')' = \tau\ \psi(\omega(z))$$

$$(-\tau)\left(\frac{e^{i\theta}-1+2}{2}\right)z^{\tau-1}(I^m(\lambda_1,\lambda_2,\ell,n)f(z))'$$

$$+z^{\tau-1}(z\ (I^m(\lambda_1,\lambda_2,\ell,n)f(z))')' = z^{\tau-1}\tau\ \psi(\omega(z))$$

$$\Longrightarrow (\tau-1)\ z^{\tau-1}\ (I^m(\lambda_1,\lambda_2,\ell,n)f(z))'$$

$$+z^{\tau-1}(z\ (I^m(\lambda_1,\lambda_2,\ell,n)f(z))')' = z^{\tau-1}\tau\ \psi(\omega(z)).$$

we thus find that

$$\left(z^{\tau-1}(z\ (I^m(\lambda_1,\lambda_2,\ell,n)f(z))')\right)' = z^{\tau-1}\tau\ \psi(\omega(z)),$$

which readily yields

$$z^{\tau-1}(z\ (I^m(\lambda_1,\lambda_2,\ell,n)f(z))') = \tau\int_0^z z^{\tau-1}\ \psi(\omega(z))\ dz$$

$$I^m(\lambda_1,\lambda_2,\ell,n)f(z) = \int_0^z \frac{\tau}{\eta^\tau}\int_0^\eta (\zeta)^{\tau-1}\psi(\omega(z))\,d\zeta\ d\eta.$$

**Theorem 3:**

Let $\psi \in \mathcal{N}$ and $\theta \in (-\pi,\pi)$. if $f \in \mathcal{R}_\theta(\psi)$,then

$$(I^m(\lambda_1,\lambda_2,\ell,n)f(z))' \prec \int_0^1 \psi\left(z\, t^{1/\tau}\right)dt \prec \psi(\omega(z)), \rightarrow (12)$$

and

$$\frac{(I^m(\lambda_1,\lambda_2,\ell,n)f(z)}{z} \prec \int_0^1 \int_0^1 \psi\left(zr\, t^{1/\tau}\right) dr\, dt \ , \rightarrow (13)$$

for all $z \in \mathbb{U}$ ,and

$$\tau = \frac{2}{1+e^{i\theta}}.$$

**Proof:**

If $f \in \mathcal{R}_\theta(\psi)$.hence from (Definition 2),in this case the differential subordination (6) hold true .

Let $p(z) = (I^m(\lambda_1,\lambda_2,\ell,n)f(z))'$ and $\tau = \frac{2}{1+e^{i\theta}}$.

Then

$$(I^m(\lambda_1,\lambda_2,\ell,n)f(z))' + \frac{1+e^{i\theta}}{2} \ z\,(I^m(\lambda_1,\lambda_2,\ell,n)f(z))'' = p(z) + \frac{1}{\tau}\, z\, p'(z) \prec \psi(z).$$

Since $Re\ (\tau) \geq 0$ and $\psi \in \mathcal{N}$ for $\theta \in (-\pi,\pi)$, and by using (Lemma 1) ,we have

$$p(z) \prec \frac{\tau}{z^\tau}\int_0^z (\zeta)^{\tau-1}\psi(\zeta))\, d\zeta \prec \psi(z)\ . \rightarrow (14)$$

With the substitution $\zeta = z\, t^{1/\tau}$ in the integral in (14) and $p(z) = (I^m(\lambda_1,\lambda_2,\ell,n)f(z))'$ the differential (14) yields

$$(I^m(\lambda_1,\lambda_2,\ell,n)f(z))' \prec \frac{\tau}{z^\tau}\int_0^1 \left(z\, t^{1/\tau}\right)^{\tau-1} \psi(z\, t^{1/\tau}))\frac{1}{\tau}\ z\, t^{\frac{1}{\tau}-1}dt \prec \phi(z) \quad .$$

$$\Rightarrow (I^m(\lambda_1,\lambda_2,\ell,n)f(z))' \prec \int_0^1 \psi\left(z\, t^{1/\gamma}\right)dt \prec \psi(z).$$

In order to obtain the differential subordination (13) ,we illustrate that the function $T(z)$given by

$$T(z) = \int_0^1 \psi\left(z\, t^{1/\tau}\right)dt\ , \rightarrow (15)$$

belongs to the class $\mathcal{N}$. To prove this we first define

$$\Phi_\tau(z) = \int_0^1 \frac{1}{1-z\,t^{1/\tau}}\, dt = \sum_{n=0}^{\infty}\frac{\tau}{n+\tau} z^n . \rightarrow (16)$$

For $Re(\tau) \geq 0$,the function $\Phi_\tau(z)$ is convex in $\mathbb{U}$ .from (16) we obtain

$$\psi(z) * \Phi_\tau(z) = \int_0^1 \frac{1}{1-z\ t^{1/\tau}}\, dt * \psi(z) = \int_0^1 \psi\left(z\, t^{1/\tau}\right)dt = T(z).$$

The convolution of two convex functions is also convex in $\mathbb{U}$ see [10]. Therefore ,the function $T(0) = 1$. Hence that $h \in \mathcal{N}$ .

Now, let

$$p(z) = \frac{I^m(\lambda_1,\lambda_2,\ell,n)f(z)}{z}.$$

$$\Rightarrow p(z) + z\, p'(z) =$$

$$\frac{I^m(\lambda_1,\lambda_2,\ell,n)f(z)}{z} + z\left(\frac{I^m(\lambda_1,\lambda_2,\ell,n)f(z)}{z}\right)'$$

$$= \frac{I^m(\lambda_1,\lambda_2,\ell,n)f(z)}{z} + z\left(\frac{z(I^m(\lambda_1,\lambda_2,\ell,n)f(z))' - I^m(\lambda_1,\lambda_2,\ell,n)f(z)}{z^2}\right)$$

$$= (I^m(\lambda_1,\lambda_2,\ell,n)f(z))'$$

Then, by using (12) and (15), we have

$$p(z) + z\, p'(z) = (\, I^m(\lambda_1,\lambda_2,\ell,n)f(z))' \prec \int_0^1 \psi\left(z\, t^{1/\tau}\right)dt = T(z).$$

By applying (Lemma 1) once more with $\tau = 1$,we obtain

$$p(z) \prec \frac{1}{z}\int_0^z T(\zeta)\,d\zeta \prec T(z). \rightarrow (17)$$

If $\zeta = rz$ substitution in the integral in (17) ,if we take into account (15) and also that

$$p(z) = \frac{I^m(\lambda_1,\lambda_2,\ell,n)f(z)}{z}$$

The first differential subordination in (17) implies that

$$\frac{I^m(\lambda_1,\lambda_2,\ell,n)f(z)}{z} \prec \frac{1}{z}\int_0^z z\ T(rz)dr \prec \int_0^1 \psi\left(z\, t^{1/\tau}\right)dt.$$

$$\Rightarrow \frac{I^m(\lambda_1,\lambda_2,\ell,n)f(z)}{z} \prec \int_0^1\int_0^1 \psi\left(z\, r\, t^{1/\tau}\right)\, dr\, dt\ .$$

**Corollary 3: [1]**

If $f \in \mathcal{R}_\theta(\psi_M), for\ all(-\pi < \theta < \pi)$, where

$$\mathcal{R}_\theta(\psi_M) = \left\{f : f \in \mathcal{A}\ and\ \left|f'(z) + \frac{1+e^{i\theta}}{2} z\, f''(z) - 1\right| \le M, (z \in \mathbb{U}; M\ > 0)\right\}.$$

and $\psi_M(z) = 1 + Mz \quad (M > 0)$. Then

$$\left|(I^m(\lambda_1,\lambda_2,\ell,n)f(z))' - 1\right| \le \frac{M\sqrt{2}}{\sqrt{5+3\cos\theta}},$$

and

$$\left|\frac{I^m(\lambda_1,\lambda_2,\ell,n)f(z)}{z} - 1\right| \le \frac{M\sqrt{2}}{2\sqrt{5+3\cos\theta}}\ .$$

There are a lot of research papers related to study integral operator and differential operator those interested in studying it can view [11], [13] , [14] [12] and [15].

## 3. Conclusion

in this work ,we have considered a certain function class $\mathcal{R}_\theta(\psi)$ of all normalized analytic functions which satisfy the followng differential subordination :

$$f'(z) + \frac{1}{2}\left(1+e^{i\theta}\right)z\, f''(z) \prec \psi(z)\ ,$$

We successfully applied of differential subordination between analytic functions, and we investigated integral representation and Convolution characterization and Differential Subordination Results.

## References


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