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A Comparison Between Some Iterative Quadrature Methods for the Numerical Solution of the Second-Kind Fredholm Integral Equations

Faoziya S. M. Musbah[a], Maryam M. A. Miftah[b] and Haniyah A. M. Saed Ben Hamdin[c]

[a]Department of Mathematics, Faculty of Education /University of Bani Waleed, Libya.
faoziyaMusbah@bwu.edu.ly

[b]Department, of Mathematics, Faculty of Education /University of Bani Waleed, Libya.
maryemmiftah77@gmail.com

[b]Department, of Mathematics, Faculty of Science /University of Sirte, Libya
h.saed1717@su.edu.ly

*Crosspnding author: faoziyaMusbah@bwu.edu.ly

Abstract: Some iterative Newton-cotes quadrature methods of closed and open types are presented here to solve the second-kind and linear Fredholm integral equations of both regular and singular kernels. The closed computational methods include the composite Simpson's 1/3 quadrature, composite trapezium-corrected Simpson's quadrature, and composite Bool's quadrature. The errors of these quadrature formulas are analyzed and estimated. A comparison between these quadrature iterative methods is carried out by solving some second-kind Fredholm integral equations of regular kernel. We achieve a good agreement between the exact and the numerical solutions of such equations, establishing the feasibility and applicability of the presented quadrature formulas. Furthermore, the composite Bool's quadrature performs better than the other two quadrature formulas. The open-type Newton-cotes quadrature methods is implemented to solve the second-kind linear Fredholm integral equations of singular kernels. A comparison between the exact and the approximate solution of such equations is carried out confirming the applicability of the method

Keywords: Composite Simpson's 1/3 quadrature; Composite trapezium-corrected Simpson's quadrature; Composite Bool's quadrature; Fredholm integral equation.

Introduction

An equation is called an integral equation if the unknown function $y(x)$ appears inside and outside the integral sign [1]. In this article, we consider the second-kind linear Fredholm Integral Equations (FIEs)

$$y(x) = f(x) + \lambda \int_a^b k(x,t) y(t) dt \tag{1}$$

where a and b are constants, $x \in [a,b]$. Assume that $f:[a,b] \to \mathbb{R}$ and $k:[a,b]\times[a,b] \to \mathbb{R}$ are continuous and that $\lambda \in \mathbb{R}$ is a regular value of $k(x,t)$, so the equation (1) is supposed to have a unique solution.

The functions f, and the so-called kernel $k(x,t)$ are given, while the function $y(x)$ is unknown, λ is a constant and non-zero parameter and its value is called eigenvalue, or characteristic value of that equation [2]. FIEs can be derived from boundary value problems with given boundary conditions [1].

FIEs have numerous applications in various fields, including potential theory, wave propagation, scattering theory, heat transfer, elasticity, and fluid flow. Many problems in science and engineering can be modeled by FIEs such as the electrostatic problem of a circular plate condenser in an unbounded perfect fluid [3]. Solving the Fredholm integral equations can be challenging, and various methods are available depending on the specific type and properties of the equation. These methods include several iterative methods using Romberg quadrature approaches that can be found in [4]. A quadrature formula was proposed in [5] to solve two-dimensional fuzzy FIEs. The error estimation was proved and the numerical stability of the method was analyzed. The method of successive approximations in terms of the midpoint quadrature formula was introduced in [6] for solving linear fuzzy FIEs of the second kind, also the numerical stability was investigated and the results were accurate. In [7] some quadrature schemes which are the repeated trapezoidal and repeated modified trapezoidal schemes were implemented to solve the second-kind linear FIEs. Shoukralla et. al. [8] presented a double-approximate numerical technique based on Legendre polynomials for solving second-kind FIEs.

In this article, we examine the performance of some computational techniques based on the quadrature formula for solving the linear and second-kind FIEs of both regular and singular kernel.

The structure of this paper is as follows: The related literature review is reviewed in the introduction section. In section 2, some closed-type iterative quadrature rules are introduced. These include the composite Simpson's 1/3 quadrature, composite trapezium-corrected Simpson's quadrature, and Composite Bool's quadrature. In section 3, an open-type iterative Newton-Cotes method is introduced to consider FIEs of singular kernel. Also, the error analysis is presented analytically. To carry out the comparison, some numerical results are shown in the results section followed by a discussion and conclusion.

2. Iterative Quadrature Rules

In this section, iterative Newton-cotes quadrature methods are presented for solving Fredholm integral equations of the second kind (1). First, let us define the partition points $x_i = a + ih,\ \ i = 1, 2, ..., n-1$ where $h = \frac{b-a}{n}$ is the step size and n is an integer. The notations y_i, f_i and k_i for the exact values of y, f and k at the point x_i.

2.1 Closed Newton-Cotes Methods

Assume that the integral term in the FIE (1) is approximated by the quadrature rule, then the FIE (1) is reduced to a system of algebraic equations as

$$y_i = f_i + \lambda \sum_{j=0}^{n} w_i k_{i,j} y_j\ ,\ \ i = 1, 2, ..., n-1$$

The weighting coefficients w_j can be computed by using Lagrange polynomials p_n of degree at most *n*. Quadrature formulas derived from Lagrange polynomial are known as Newton-Cotes formulas [13].

2.1.1 Composite Simpson's 1/3 Quadrature Method

The composite Simpson's 1/3 formula with two adjacent strips is given by the form [9]

$$\int_a^b f(x)dx \cong \frac{h}{3}\left(f(x_0) + 2\sum_{j=1}^{\frac{n}{2}-1} f(x_{2j}) + 4\sum_{j=1}^{\frac{n}{2}} f(x_{2j-1}) + f(x_n) \right) \tag{2}$$

The truncation error term E_S of this formula has the form

$$E_S = \frac{(b-a)h^4}{180} f^{(4)}(\xi),\quad \xi \in [a,b]$$

The truncation error of the composite Simpson's 1/3 method is of the order $O(h^4)$ and the degree of precision is $n = 3$. Note that, the formula can be applied only to integrals with equal intervals of even numbers.

Approximating the solution of the FIE (1) by using the formula (2) gives

$$y_i = f_i + \frac{\lambda h}{3}\left(k_{i,0}y_0 + 2\sum_{j=1}^{\frac{n}{2}-1} k_{i,2j}y_{2j} + 4\sum_{j=1}^{\frac{n}{2}} k_{i,2j-1}y_{2j-1} + k_{i,n}y_n \right)$$

$, \; i = 1,2,\ldots,n-1$ (3)

The matrix form of the above system can be displayed as

$$\frac{\lambda h}{3}\begin{pmatrix} 4k_{1,1}-\frac{3}{h} & 2k_{1,2} & 4k_{1,3} & \cdots & 4k_{1,n-1} \\ 4k_{2,1} & 2k_{2,2}-\frac{3}{h} & 4k_{2,3} & \cdots & 4k_{2,n-1} \\ 4k_{3,1} & 2k_{3,2} & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 4k_{n-1,n-1} \\ 4k_{n-1,1} & 2k_{n-1,2} & 4k_{n-1,3} & \cdots & 4k_{n-1,n-1}-\frac{3}{h} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} -f_1 - \frac{\lambda h}{3}\left(k_{1,0}y_0 + k_{1,n}y_n\right) \\ -f_2 - \frac{\lambda h}{3}\left(k_{2,0}y_0 + k_{2,n}y_n\right) \\ -f_3 - \frac{\lambda h}{3}\left(k_{3,0}y_0 + k_{3,n}y_n\right) \\ \vdots \\ -f_{n-1} - \frac{\lambda h}{3}\left(k_{n-1,0}y_0 + k_{n-1,n}y_n\right) \end{pmatrix}$$

2.1.2 Composite Trapezium-Corrected Simpson's Method

The composite trapezium-corrected Simpson's (TCS) formula with equal strips of odd numbers is addressed in the form [10]

$$\int_a^b f(x)dx \cong \frac{h}{6}\left(2f(x_0) + 4\sum_{j=1}^{\frac{n-3}{2}} f(x_{2j}) + 8\sum_{j=1}^{\frac{n-1}{2}} f(x_{2j-1}) \right.$$

$$\left. +5f(x_{n-1}) + 3f(x_n) \right) \quad (4)$$

where n must be odd. The truncation error term E_{TCS} of the above formula is

$$E_{TCS} = \frac{(b-a)h^2}{12} f''(\xi) + \frac{(b-a)h^4}{180} f^{(4)}(\xi), \;\; \xi \in [a,b]$$

The truncation error of the composite trapezium-corrected Simpson's formula is of the order $O(h^2)$. Approximating the solution of the FIE (1) by using the formula (4) gives

$$y_i = f_i + \frac{\lambda h}{6}\left(2k_{i,0}y_0 + 4\sum_{j=1}^{\frac{n-3}{2}} k_{i,2j}y_{2j} + 8\sum_{j=1}^{\frac{n-1}{2}} k_{i,2j-1}y_{2j-1}, \right.$$

$$\left. +5k_{i,n-1}y_{n-1} + 3k_{i,n}y_n \right), \qquad i = 1,2,\ldots,n-1 \;\; (5)$$

The matrix form of the above system can be displayed as

$$\frac{\lambda h}{6}\begin{pmatrix} 8k_{1,1}-\frac{6}{h} & 4k_{1,2} & 8k_{1,3} & \cdots & 5k_{1,n-1} \\ 8k_{2,1} & 4k_{2,2}-\frac{6}{h} & 8k_{2,3} & \cdots & 5k_{2,n-1} \\ 8k_{3,1} & 4k_{3,2} & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 5k_{n-1,n-1} \\ 8k_{n-1,1} & 4k_{n-1,2} & 8k_{n-1,3} & \cdots & 4k_{n-1,n-1}-\frac{6}{h} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} -f_1 - \frac{\lambda h}{6}\left(2k_{1,0}y_0 + 3k_{1,n}y_n\right) \\ -f_2 - \frac{\lambda h}{6}\left(2k_{2,0}y_0 + 3k_{2,n}y_n\right) \\ -f_3 - \frac{\lambda h}{6}\left(2k_{3,0}y_0 + 3k_{3,n}y_n\right) \\ \vdots \\ -f_{n-1} - \frac{\lambda h}{6}\left(2k_{n-1,0}y_0 + 3k_{n-1,n}y_n\right) \end{pmatrix}$$

2.1.3 Composite Bool's Quadrature Method

The composite Bool's (for n to be a multiple of 4) [11, 12] is given as follows:

$$\int_a^b f(x)dx \cong \frac{2h}{45}\sum_{j=1}^{\frac{n}{4}}\left(7f(x_{4j-4}) + 32f(x_{4j-3}) \right.$$

$$\left. +12f(x_{4j-2}) + 32f(x_{4j-1}) + 7f(x_{4j}) \right) \quad (6)$$

where $n = 4k, k \geq 1$. The truncation error term E_B of this formula is

$$E_B = -\frac{2(b-a)h^6}{945} f^{(6)}(\xi) - \frac{(b-a)h^8}{450} f^{(8)}(\xi), \; \xi \in [a,b]$$

This method has a degree of precision $n = 5$ and the truncation error is of the order $O(h^6)$. Generally, the degree of precision of the n-point Newton-cotes quadrature formula is $(n-1)$ when n is even and n when n is odd [12].

Approximating the solution of the FIE (1) by using the formula (6) gives

$$y_i = f_i + \frac{2\lambda h}{45}\sum_{j=1}^{\frac{n}{4}}\Big(7k_{i,4j-4}y_{4j-4} + 32k_{i,4j-3}y_{4j-3} + 12k_{i,4j-2}y_{4j-2}$$

$$+32k_{i,4j-1}y_{4j-1} + 7k_{i,4j}y_{4j}\Big), \quad i = 1, 2, \ldots, n-1 \quad (7)$$

The matrix form of the above system can be displayed as

$$\frac{2\lambda h}{45}\begin{pmatrix} 32k_{1,1}-\frac{45}{2h} & 12k_{1,2} & 32k_{1,3} & 14k_{1,4} & \cdots & 32k_{1,n-2} \\ 32k_{2,1} & 12k_{2,2}-\frac{45}{2h} & 32k_{2,3} & 14k_{2,4} & \cdots & 32k_{2,n-2} \\ 32k_{3,1} & 12k_{3,2} & 32k_{3,3}-\frac{45}{2h} & 14k_{3,4} & \cdots & 32k_{3,n-2} \\ 32k_{4,1} & 12k_{4,2} & 32k_{4,3} & 14k_{4,4}-\frac{45}{2h} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 32k_{n-1,n-2} \\ 32k_{n-1,1} & 12k_{n-1,2} & 32k_{n-1,3} & 14k_{n-1,4} & \cdots & 32k_{n-1,n-1}-\frac{45}{2h} \end{pmatrix} \times$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} -f_1 - \frac{14\lambda h}{45}\left(k_{1,0}y_0 + k_{1,n}y_n\right) \\ -f_2 - \frac{14\lambda h}{45}\left(k_{2,0}y_0 + k_{2,n}y_n\right) \\ -f_3 - \frac{14\lambda h}{45}\left(k_{3,0}y_0 + k_{3,n}y_n\right) \\ -f_4 - \frac{14\lambda h}{45}\left(k_{4,0}y_0 + k_{4,n}y_n\right) \\ \vdots \\ -f_{n-1} - \frac{14\lambda h}{45}\left(k_{n-1,0}y_0 + k_{n-1,n}y_n\right) \end{pmatrix}$$

Equations (3), (5), and (7) give the approximate solution of the second kind linear FIE (1) at $y_i = y(x_i)$.

2.2 Open Newton-Cotes Methods

The open Newton-Cotes formulas are given in the following form [13]:

$$\int_a^b p_n(x)dx = h\int_{-1}^{n+1}\sum_{j=0}^{n}\Delta^j f_0\binom{s}{j}ds, \quad \binom{s}{j} = \frac{s!}{j!(s-j)!} \quad (8)$$

where $\Delta^j f_0 = \Delta(\Delta^{j-1} f_0)$ is called forward differences of the order j, and $s = \frac{x-x_0}{h}$,

$h = \frac{b-a}{n+2}$, $x_0 = a + h$, $x_n = b - h$, $x_i = x_0 + ih$.

Now approximating the solution of the second-kind FIE (1) by using the above approximation, one obtains

$$y_i = f_i + \lambda h\int_{-1}^{n+1}\sum_{j=0}^{n}\Delta^j k_{i,0}y_0\binom{s}{j}ds, \quad i = 0, 1, \ldots, n \quad (9)$$

It should be noticed that the difference between closed and open Newton-Cotes formulas is that the approximating polynomial in closed Newton-Cotes formulas interpolates the internal points as well as the endpoints a and b while in open Newton-Cotes formulas, the approximating polynomial interpolates only the points between endpoints a and b.

3. Numerical Results and Discussion

In this section, some numerical examples are given to demonstrate our theoretical results. All computational results shown here are carried out using the Mathematica Wolfram 13.1.

3.1 Closed Newton-Cotes methods

For verification purposes, we will implement the iterative quadrature rules (3), (5), and (7), to find the numerical solutions of some the second kind linear FIEs.

Example 1. Let us consider the following second-kind linear FIE [7]:

$$y(x) = x + \int_0^1 (4xt - x^2)y(t)dt, \quad x \in [0,1] \quad (10)$$

The exact solution of the FIE (10) is

$$y(x) = (24x - 9x^2).$$

The results of the errors of the methods for FIE (10) of the second kind in discrete l^∞ and l^2 norm are listed in Table 1. The following formulas evaluate the maximum error and l^2 error: $\|e\|_{l^\infty} = \underset{1\le i\le n}{Max}|e_i|$ and $\|e\|_{l^2} = \left(\sum_{i=1}^{n-1} h|e_i|^2\right)^{1/2}$.

Table 1: l^∞ and l^2 errors of the quadrature methods for the FIE (10).

error	Bool (n = 40)	Trapezium-corrected Simpson (n = 41)	Simpson (n = 40)
$\|e\|_{l^\infty}$	5.32907E-15	7.94733E-5	8.88178E-15
$\|e\|_{l^2}$	2.33470E-15	5.64095E-5	3.06128E-15

The absolute errors (absolute difference between the exact and the numerical solution) of the computational methods are illustrated in Fig.1, Fig. 2, and Fig. 3.

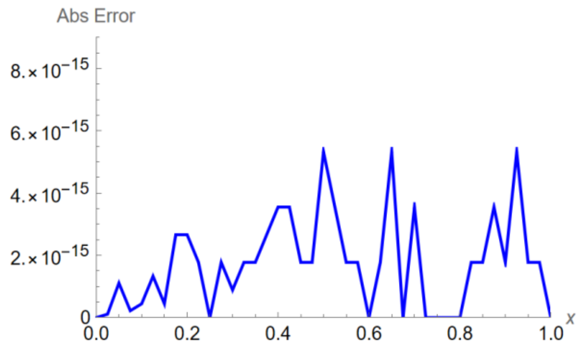

Fig. 1: Absolute errors of Bool's method for $n = 40$.

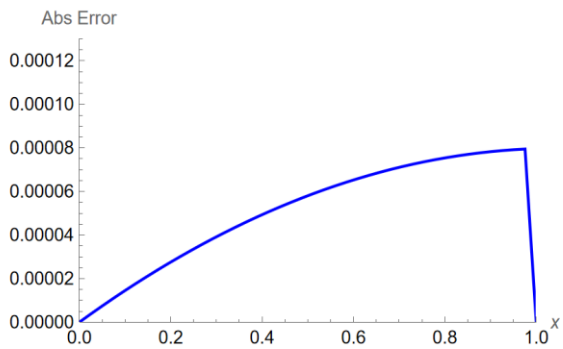

Fig. 2: Absolute errors of trapezium-corrected Simpson's method for $n = 41$.

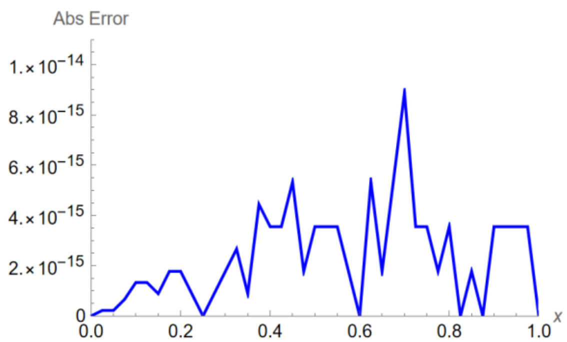

Fig. 3: Absolute errors of Simpson's method for $n = 40$.

Example 2. Consider the following second-kind linear FIE [8]:

$$y(x) = e^{-x} + \int_0^1 e^{x+t} y(t)dt \ , \quad x \in [0,1] \qquad (11)$$

The exact solution of the FIE (11) is

$$y(x) = e^{-x} + \frac{2e^x}{3-e^2} .$$

The maximum error and l^2 errors of the computational schemes are presented in Table 2.

Table 2: l^∞ and l^2 errors of the quadrature methods for the FIE (11).

error	Bool ($n = 40$)	Trapezium-corrected Simpson ($n = 41$)	Simpson ($n = 40$)
$\|e\|_{l^\infty}$	5.98575E-11	2.01666E-5	6.30521E-8
$\|e\|_{l^2}$	3.96902E-11	1.33694E-5	4.18086E-8

The absolute errors of the computational methods are illustrated in Fig.4, Fig. 5, and Fig. 6.

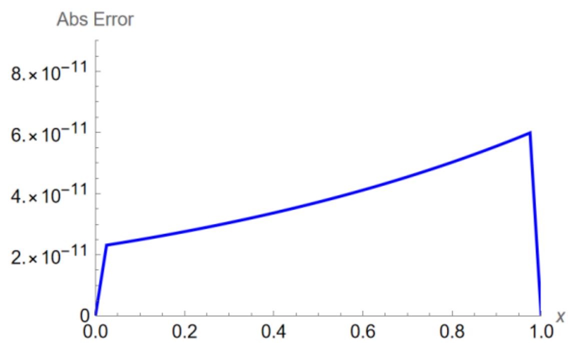

Fig. 4: Absolute errors of Bool's method for $n = 40$.

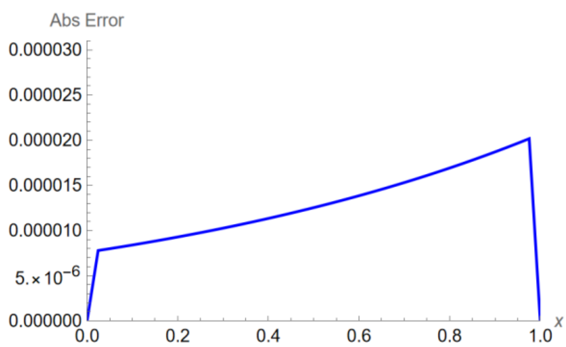

Fig. 5: Absolute errors of trapezium-corrected Simpson's method for $n = 41$.

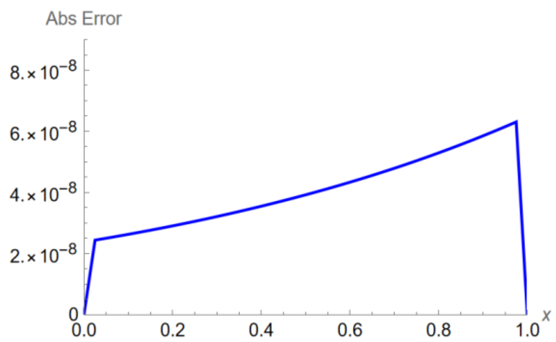

Fig. 6: Absolute errors of Simpson's method for $n = 40$.

Example 3. Consider the following second-kind linear FIE [14]:

$$y(x) = e^{x} + \tfrac{1}{2}\int_{-1}^{1} |x-t| y(t)\,dt\,, \quad x \in [-1,1] \qquad (12)$$

The exact solution of the FIE (12) is

$$y(x) = \frac{1}{2} x e^{x} + \left(\frac{1+e^4+6e^2}{8(1+e^2)} + \frac{1}{1+e^2} \right) e^{x} + \left(\frac{1+e^4+6e^2}{8(1+e^2)} \right) e^{-x}.$$

The maximum error and l^2 errors of the computational schemes are listed in Table 3.

Table 3: l^∞ and l^2 errors of the quadrature methods for the FIE (12).

error	Bool ($n = 40$)	Trapezium-corrected Simpson ($n = 41$)	Simpson ($n = 40$)
$\|e\|_{l^\infty}$	6.34174E-3	4.50952E-3	5.50306E-3
$\|e\|_{l^2}$	5.02603E-3	3.57334E-3	4.44440E-3

The absolute errors of the computational methods are presented in Fig. 7, Fig. 8, and Fig. 9.

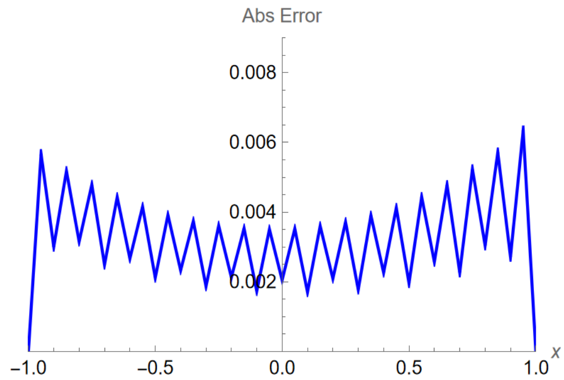

Fig. 7: Absolute errors of Bool's method for $n = 40$.

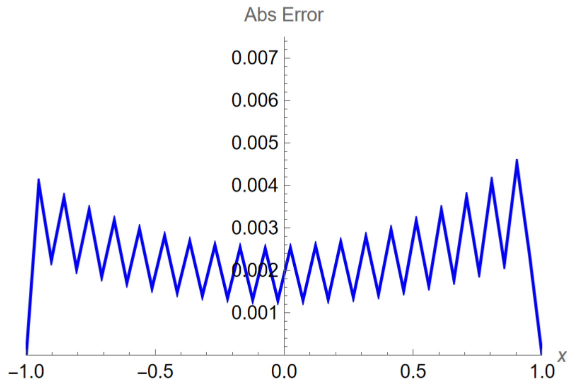

Fig. 8: Absolute errors of trapezium-corrected Simpson's method for $n = 41$.

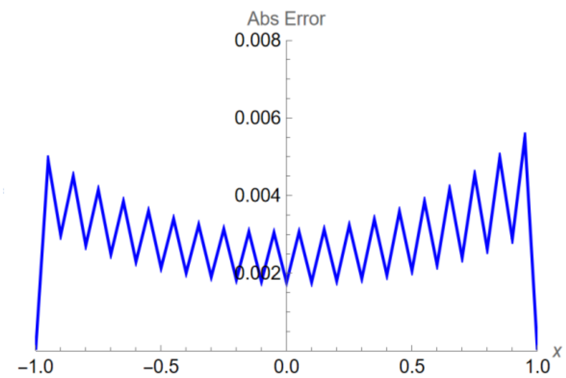

Fig. 9: Absolute errors of Simpson's method for $n = 40$.

3.2 Open Newton-Cotes methods

Here we will present some illustrative examples of linear, second-kind FIEs with singular kernels solved by implementing the open Newton-Cotes method (8).

Example 4. Consider the following second-kind linear FIE with a singular kernel:

$$y(x)=\sqrt[3]{x}-\frac{\sqrt{\pi}\,\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{11}{6}\right)}+\int_0^1 \frac{1}{\sqrt{1-t}}y(t)dt\ ,\quad x\in[0,1] \qquad (13)$$

The exact solution of the FIE (13) is

$$y(x)=\sqrt[3]{x}\ .$$

The maximum error and l^2 errors of the computational schemes are presented in Table 4.

Table 4: l^∞ and l^2 errors of the quadrature methods for the FIE (13).

error	open Newton-Cotes $(n=30)$
$\Vert e\Vert_{l^\infty}$	1.07630E-1
$\Vert e\Vert_{l^2}$	1.95935E-1

Fig. 10 shows a comparison between the exact solution and the approximate solution obtained by the open Newton-Cotes for the FIE (13).

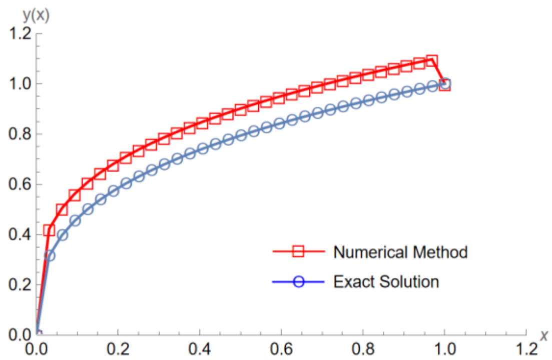

Fig. 10: Comparison between the exact and approximate solutions obtained by the open Newton-Cotes for $n=30$.

Example 5. Consider the following second-kind linear FIE with a singular kernel:

$$y(x)=(x-2)^2-\frac{56}{12}+\int_0^1 \frac{1}{\sqrt{1-t}}y(t)dt\ ,\quad x\in[0,1] \qquad (14)$$

The exact solution of the FIE (14) is

$$y(x)=(x-2)^2\ .$$

The maximum error and l^2 errors of the computational schemes are presented in Table 5.

Table 5: l^∞ and l^2 errors of the quadrature methods for the FIE (14).

error	open Newton-Cotes $(n=30)$
$\Vert e\Vert_{l^\infty}$	1.07609E-1
$\Vert e\Vert_{l^2}$	1.05591E-1

Fig. 11 compares the computational open Newton-Cotes method results with the exact solution for the FIE (14).

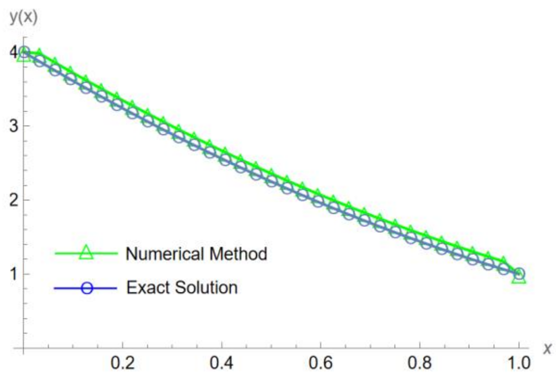

Fig. 11: Comparison between the exact and approximate solutions obtained by the open Newton-Cotes for $n=30$.

We can see from Figures 10 and 11 that the computational solutions of the open Newton-Cotes method (8) and the exact solutions are in good agreement.

Conclusion

To conclude, three closed-type and iterative Newton-Cotes methods were implemented to solve linear and second-kind FIEs of regular kernels. The errors of these iterative methods were estimated in discrete l^∞ and l^2 norm confirming their efficiency. Respectively, Figures 1 and 4 show that the absolute difference between the exact and the computational solutions obtained by the composite Bool's quadrature are of order 10^{-15} and 10^{-11}. Thus, one could claim that the performance of the composite Bool's quadrature is much better than the other two iterative Newton-Cotes methods. Moreover, the open Newton-Cotes formula (9) was implemented to solve linear and second-kind FIEs of singular kernels. The comparison of the obtained approximate solutions against the exact solution shows an acceptable agreement as shown in Figures 10 and 11.

Arabic section:

العنوان:

مقارنة بين بعض الطرق التربيعة التكرارية للحل العددي لمعادلات فريدهولم التكاملية الخطية من النوع الثاني

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د. فوزية صالح محمود مصباح

أ. مريم مفتاح علي مفتاح

د. هنية سعد بن حمدين

الملخص:

بعض طرق نيوتن-كوتس التربيعة التكرارية المفتوحة والمغلقة تم عرضها في هذا البحث لحل معادلات فريدهولم التكاملية الخطية من النوع الثاني ذات الانوية الشادة والغير شادة. الطرق العددية المغلقة تتضمن تربيعة 1/3 سمبسون المركبة، شبه المنحرف-سمبسون المصححة المركبة وتربيعة بول المركبة. تم تحليل وتقييم أخطاء هذه الطرق. بعض الأمثلة عن معادلات فريدهولم التكاملية الخطية من النوع الثاني ذات الانوية الغير شادة تم حلها بهذه الطرق، لنبين مدى سهولة وقابلية هذه الطرق للحل. من المقارنة نجد أن طريقة بول المركبة تعطي نتائج أفضل من الطرق الأخرى المغلقة. طرق التربيعية لنيوتن-كوتس المفتوحة تمت توظيفها لحل معادلات فريدهولم التكاملية الخطية من النوع الثاني ذات الانوية الشادة ومقارنة النتائج العددية مع الحل التحليلي.

الكلمات المفتاحية:

تربيعة 1/3 سمبسون المركبة، شبه المنحرف-سمبسون المصححة المركبة، تربيعة بول المركبة، معادلة فريدهولم التكاملية.

Abbreviations and Acronyms

FIEs: Fredholm integral equations.

TCS: trapezium-corrected Simpson's.

Duality of interest

The authors declare that they have no duality of interest associated with this manuscript.

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