المؤتمر العلمي الدولي الثالث للهندسة والعلوم

The 3RD Scientific International Conference in Engineering & Science

http://bwu.edu.ly/icse2024

Received 25/07/2024 Revised 18/08/2024 Published 10/09/2024

icse@bwu.edu.ly

Explicit Solutions for the Nonlinear Pochhammer-Chree Equation Utilizing Enhanced Tanh-Function Expansion Scheme and a Direct Approach

Khaled A. Alurrfi[1], Mustafa A. Shibah[2*] and AML M. Abraheem[3]

[1] Department of Mathematics, Faculty of Science, Elmergib University, Khoms, Libya.

[2*] Department of Mathematics, Faculty of Education, Wadi-Alshatti, Brak, Libya.

[3] Department of Mathematical Sciences, The Libyan Academy, Tripoli, Libya.

E-mail addresses: alurrfi@yahoo.com, m.shibah@wau.edu.ly and 21981606@academy.edu.ly

*Corresponding author: m.shibah@wau.edu.ly (Mustafa A. Shibah)

Abstract:

In this paper, we will apply the tanh-function scheme with the aid of Maple software to construct new explicit wave solutions of the nonlinear Pochhammer-Chree equation. Also, we use a direct algebraic method based on the Liénard equation to find other diffrent new explicit solutions. Soliton solutions, periodic solutions and rational functions solutions are obtained. Comparing our new results obtained in this paper with the well-known results are given. Further, some 2D and 3D graphs of the obtained explicit traveling wave solutions are shown. Finally, the tanh-function expansion scheme presented in this paper is straightforward, concise and it can also be applied to other nonlinear partial differential equations in mathematical physics.

Keywords: Explicit wave solutions, nonlinear PDEs, nonlinear Pochhammer-Chree equation, the tanh-function scheme, Liénard equation.

1. Introduction

In the recent years, many new natural phenomena exist in mathematical physics and some other fields as plasma physics, biology, chemistry, engineering, quantum mechanics, fluid mechanics, optical fibers, hydrodynamic waves and etc., which can be describe by nonlinear PDEs.

There are many analytical schemes to obtain explicit wave solutions for the nonlinear PDEs in mathematics and physics such as the modified extended tanh-function scheme [1-5], the $\left(\frac{G'}{G}\right)$expansion approach [6-9], the generalized $\left(\frac{G'}{G},\frac{1}{G}\right)$expansion method [10], the Exp-function method [11], the Jacobi-elliptic function method [12], the auxiliary equation method [13], the generalized-projective Riccati equations method [14], the modified algebraic method [15], the new mapping method [16] and etc.

The tanh-function scheme depends on adding integration constants to the resulting nonlinear ODEs from the nonlinear PDEs using wave transformation.

The objective of this article is to employ the modified tanh-function expansion scheme [1, 2] and the direct approach with the help of the Liénard equation [10], for finding new explicit wave solutions of the following nonlinear Pochhammer-Chree equation [17]:

$$u_{tt} - u_{xxtt} - (\vartheta u + \beta u^3 + \gamma u^5)_{xx} = 0, \qquad (1.1)$$

where ϑ, β, γ are constants.

The Eq. (1.1) represents nonlinear models of longitudinal wave propagation in elastic rods and it has discussed in [17] by using the (G'/G,1/G) expansion method.

This article is organized as follows. In section 2, we give the description of the modified tanh-function expansion scheme. In section 3, we apply this method to the nonlinear Pochhammer-Chree equation. In section 4, further results for the nonlinear Pochhammer-Chree are obtained. In section 5, conclusions are given.

2. Description of the modified tanh-function expansion method

We suppose that the given nonlinear partial differential equation for $u(x,t)$ to be in the form:

$$P(u,\, u_x, u_t,\, u_{xx}, u_{xt}, u_{tt}, \dots) = 0, \qquad (2.1)$$

where P is a polynomial in its arguments. The essence of the modified tanh-function expansion method can be presented in the following steps [1, 2]:

Step 1: Seek travelling wave solutions of Eq. (2.1) by taking

$$u(x,t) = u(\xi), \qquad \xi = x - ct\,, \qquad (2.2)$$

The transformation (2.2) converts Eq. (2.1) to the ordinary differential equation (ODE):

$$Q(u, u', u'', \dots) = 0, \qquad (2.3)$$

where prime denotes the derivative with respect to ξ.

Step 2: If possible, integrate Eq. (2.3) term by term one or more times. This yields constant(s) of integration. For simplicity, the integral constant(s) may be zero.

Step 3: We assume that Eq. (2.3) has the formula solution:

$$u(\xi) = a_0 + \sum_{i=1}^{n} a_i \phi^i(\xi) + \sum_{i=1}^{n} b_i \phi^{-i}(\xi), \qquad (2.4)$$

where n is a positive integer that can be determined by balancing the highest-order derivative term with the highest nonlinear term in Eq. (2.4), $a_0, a_i, b_i,\ i = 1,2,\dots,n$ are parameters to be determined such that $a_n \neq 0$ or $b_n \neq 0$ and $\phi'(\xi)$ is a solution of the following Riccati equation:

$$\phi'(\xi) = b + \phi^2(\xi), \qquad (2.5)$$

where b is a constant. It is well-known that Eq. (2.5) has three types of explicit solutions [1, 2]. In some nonlinear equations the balance number n is not a positive integer. In this case, we make the following transformations [17]:

(a) when $n = q/p$, where q/p is a fraction in the lowest terms, we let

$$u(\xi) = v^{\frac{q}{p}}(\xi)\,, \qquad (2.6)$$

then substitute (2.6) into (2.3) to get a new equation in the new function $v(\xi)$ with a positive integer balance number;

(b) when n is a negative number, we let

$$u(\xi) = v^n(\xi), \qquad (2.7)$$

and substitute (2.7) into (2.3) to get a new equation in the new function $v(\xi)$ with a positive integer balance number.

Step 4: We Substitute (2.4) with (2.5) into Eq. (2.3) yields a set of algebraic equations involving $a_0, a_i, b_i,\ i = 1,2,\dots,n$ and c, which can be solved using Maple or Mathematica to obtain analytic explicit solutions of the nonlinear PDE (2.1) in closed form. In the next sections, we will find the explicit solutions of Eq. (1.1) using the modified tanh-function expansion method and a direct method with the help of Lienard equation.

3. Explicit wave solutions of Eq. (1.1) using the modified tanh-function expansion method

In this section, we will apply the modified tanh-function expansion method to construct new explicit solutions of the nonlinear Pochhammer-Chree equation (1.1). To this aim, we use the wave transform (2.2) to convert Eq. (1.1) to the following nonlinear ODE:

$$c^2u'' - c^2u'''' - (\vartheta u + \beta u^3 + \gamma u^5)'' = 0. \quad (3.1)$$

Integrating (3.1) w. r. to ξ twice, we have

$$(\vartheta - c^2)u + c^2u'' + \beta u^3 + \gamma u^5 = 0. \quad (3.2)$$

By balancing u^5 with u'' in Eq. (3.2), we get $n = \frac{1}{2}$. Therefore, we use the new transformation:

$$u(\xi) = v^{\frac{1}{2}}(\xi), \quad (3.3)$$

where $v(\xi)$ is a new function of ξ . Substituting (3.3) into Eq. (3.2) ,we get the new nonlinear ODE:

$$(\vartheta - c^2)v^2 + \frac{c^2}{4}(2vv'' - (v')^2) + \beta v^3 + \gamma v^4 = 0, \quad (3.4)$$

we balance the variables vv'' with v^4 in Eq. (3.4) giving $N = 1$. Thus we obtain the corresponding solution:

$$v(\xi) = a^0 + a_1\phi + b_1\phi^{-1}, \quad (3.5)$$

where a_0, a_1, b_1 are constants to be detemined, such that $a_1 \neq 0$, or $b_1 \neq 0$, while ϕ satisfies the Riccati Eq. (1.5). Substituting (3.5) into (3.4) and using (1.5), the left-hand side of (3.4) becomes a polynomial in ϕ. Setting the coefficients of this polynomial to be zero yields a system of algebraic equations as follows:

$\phi^4: a_1^4\gamma + \frac{3}{4}a_1^2c^2 = 0,$

$\phi^3: a_0a_1c^2 + a_1^3\beta + 4a_0a_1^3\gamma = 0,$

$\phi^2: a_1^2\vartheta - a_1^2c^2 + \frac{1}{2}a_1^2bc^2 + \frac{3}{2}a^1b^1c^2 + 3a^{0a_1^2}\beta +$
$6a_0^2a_1^2\gamma + 4a_1^3b_1\gamma = 0,$

$\phi: 2a_0a_1\vartheta - 2a_0a_1c^2 + a_0a_1bc^2 + 3a_0^2a_1\beta + 3a_1^2b_1\beta +$
$4a_0^3a_1\gamma + 12a_0a_1^2b_1\gamma = 0,$

$\phi^0: a_0^2\vartheta + 2a_1b_1\vartheta - a_0^2c^2 - 2a_1b_1c^2 + 3a_1b_1bc^2 -$
$\frac{1}{4}c^2a_1^2b^2 - \frac{1}{4}c^2b_1^2 + a_0^3\beta + 6a_0a_1b_1\beta + a_0^4\gamma +$
$a_0^2a_1b_1\gamma + 6a_1^2b_1^2\gamma = 0,$

$\phi^{-1}: 2a^0b^1\vartheta - 2a^0b^1c^2 + a^0b^1bc^2 + 3a_0^2b^1\beta +$
$3a_1b_1^2\beta + 4a_0^3b_1\gamma + 12a_0a_1b_1^2\gamma = 0,$

$\phi^{-2}: b_1^2\vartheta - b_1^2c^2 + \frac{3}{2}a_1b_1b^2c^2 + \frac{1}{2}b_1^2bc^2 + 3a_0b_1^2\beta +$
$6a_0^2b_1^2\gamma + 4a_1b_1^3\gamma = 0,$

$\phi^{-3}: a_0b_1b^2c^2 + b_1^3\beta + 4a_0b_1^3\gamma = 0,$

$\phi^{-4}: \frac{3}{4}b_1^2b^2c^2 + b_1^4\gamma = 0.$

On solving the above algebraic equations using Maple, we get the following results:

Case 1.

$$b = \frac{-(c^2 - \vartheta)}{c^2}, a_0 = \pm\sqrt{\frac{-3(c^2 - \vartheta)}{4\gamma}}, \quad a_1 = \pm\sqrt{\frac{3c^2}{-4\gamma}},$$

$$b_1 = 0, \qquad \beta = \pm\frac{8}{3}\sqrt{\frac{-3(c^2 - \vartheta)}{4\gamma}}, \quad (3.6)$$

where $\gamma < 0$ and $(c^2 - \vartheta) > 0$.

From (3.3), (3.5) and (3.6), we can derive the explicit wave solutions as follows:

(I) For $c^2 - \vartheta > 0, \Rightarrow\ b < 0$, then we have the dark soliton solution:

$$u(\xi) = \left[\sqrt{\frac{-3(c^2 - \vartheta)}{4\gamma}}\left(1 \pm \tanh\left(\sqrt{\frac{c^2 - \vartheta}{c^2}}\xi\right)\right)\right]^{\frac{1}{2}}, \quad (3.7)$$

where $\xi = x - ct$.

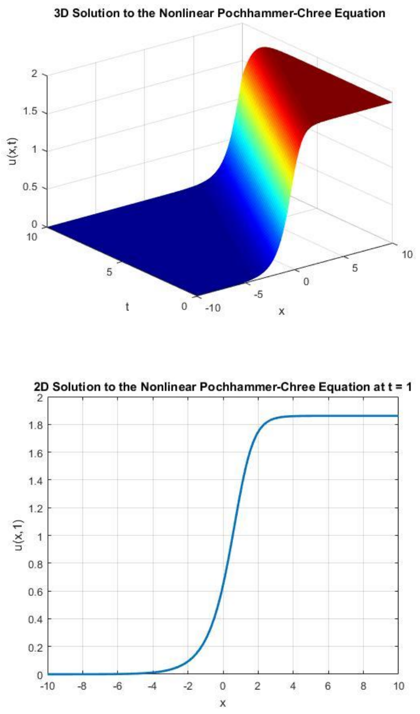


Figure 1. The profile of the dark-soliton solution (3.7) with $c = 1,\ \alpha = \frac{1}{2}, \gamma = -2$.

(II) For $c^2 - \alpha = 0, \Rightarrow b = 0,\ \ c = \pm\sqrt{a},\ a_0 = 0$, and $a_1 = \pm\sqrt{\frac{-3\vartheta}{4\gamma}}$, then we have the solution:

$$u(\xi) = \left[\mp\sqrt{\frac{-3\vartheta}{4\gamma}}\frac{1}{\xi}\right]^{\frac{1}{2}}, \tag{3.8}$$

where $\xi = x \mp \sqrt{\vartheta}t$.

Case 2.

$$b = \frac{\vartheta - c^2}{c^2},\ \ a_0 = \pm\sqrt{\frac{3(\vartheta - c^2)}{4\gamma}},\ \ a_1 = 0,$$

$$b_1 = \pm\sqrt{\frac{3}{-4\gamma}\left(\frac{\vartheta - c^2}{c^2}\right)},$$

$$\beta = \pm\frac{4}{3}\sqrt{\frac{3(\vartheta - c^2)}{\gamma}},\quad c = c, \tag{3.9}$$

where $\gamma > 0$ and $(\vartheta - c^2) > 0$.

If For $\vartheta - c^2 < 0, \Rightarrow\ b < 0$, then we have the singular soliton solution:

$$u(\xi) = \left[\sqrt{\frac{-3(c^2 - \vartheta)}{4\gamma}}\left(1 \pm \coth\left(\sqrt{\frac{c^2 - \vartheta}{c^2}}\xi\right)\right)\right]^{\frac{1}{2}}, \tag{3.10}$$

where $\xi = x - ct$.

Case 3.

$$b = \frac{c^2 - \vartheta}{5c^2},\qquad a_0 = \pm\frac{1}{5}\sqrt{\frac{-15(c^2 - \vartheta)}{\gamma}},$$

$$a_1 = \pm\sqrt{\frac{3c^2}{-4\gamma}},\ \ b_1 = \pm\sqrt{\frac{3(c^2 - \vartheta)^2}{-100\gamma c^2}},$$

$$\beta = \pm\frac{8}{3}\sqrt{\frac{-3\gamma(c^2 - \vartheta)}{5}},\qquad c = c, \tag{3.11}$$

where $\gamma < 0$ and $(c^2 - \vartheta) > 0$.

(I) For $c^2 - \vartheta > 0, \Rightarrow b > 0$, we have the periodic-solution

$$u(\xi) = \left[\frac{1}{5}\sqrt{\frac{-15(c^2 - \vartheta)}{\gamma}}\left(1 \pm \frac{1}{2}\tan\left(\sqrt{\frac{c^2 - \vartheta}{5c^2}}\xi\right) \pm \frac{1}{2}\cot\left(\sqrt{\frac{c^2 - \vartheta}{5c^2}}\xi\right)\right)\right]^{\frac{1}{2}}, \tag{3.12}$$

where $\xi = x - ct$.

(II) If $b = 0$, then we obtain the same solution (3.10)

Case 4.

$$b = \frac{-(c^2 - \vartheta)}{4c^2},\qquad a_0 = \pm\sqrt{\frac{-3(c^2 - \vartheta)}{4\gamma}},$$

$$a_1 = \pm\sqrt{\frac{-3c^2}{4\gamma}},\quad b_1 = \pm\sqrt{\frac{-3(c^2-\vartheta)^2}{64\gamma c^2}},$$

$$\beta = \pm\frac{8}{3}\sqrt{\frac{-3\gamma(c^2-\vartheta)}{4}},\qquad c = c, \qquad (3.13)$$

where $\gamma < 0$ and $(c^2-\vartheta) > 0$.

(I) For $c^2-\vartheta > 0,\ \Rightarrow\ b < 0$, then we have the straddled solitons solution:

$$u(\xi) = \left[\sqrt{\frac{-3(c^2-\vartheta)}{4\gamma}}\left(1 \pm \frac{1}{2}\tanh\left(\sqrt{\frac{c^2-\vartheta}{4c^2}}\xi\right) \pm \frac{1}{2}\coth\left(\sqrt{\frac{c^2-\vartheta}{4c^2}}\xi\right)\right)\right]^{\frac{1}{2}}, \quad (3.14)$$

where $\xi = x - ct$.

(II) If $b = 0$, then we get the same solution (3.10)

Case 5.

$$b = \frac{-(4\gamma a^{12} + 3\vartheta)}{4\gamma a^{12}},\quad a_0 = 0,\quad a_1 = \pm\sqrt{\frac{-3c^2}{4\gamma}},$$

$$b_1 = \frac{-(4\gamma a^{12} + 3\vartheta)}{4\gamma a^{1}},\quad \beta = 0,$$

$$c = \pm\sqrt{\frac{-4\gamma {a_1}^2}{3}}, \qquad (3.15)$$

where $\gamma < 0$.

(I) For $\gamma(4\gamma {a_1}^2 + 3\vartheta) > 0, \Rightarrow b < 0$, then we have the straddled solitons solution:

$$u(\xi) = \left[\sqrt{\frac{3(\vartheta-c^2)}{4\gamma}}\left(\tanh\left(\sqrt{\frac{-(\vartheta-c^2)}{c^2}}\xi\right) - \coth\left(\sqrt{\frac{-(\vartheta-c^2)}{c^2}}\xi\right)\right)\right]^{\frac{1}{2}},$$

(3.16)

Where

$$\xi = x \mp \sqrt{\frac{-4\gamma}{3}}a_1 t.$$

(II) For $\gamma(4\gamma {a_1}^2 + 3\vartheta) < 0, \Rightarrow b > 0$, then we have the periodic solution:

$$u(\xi) = \left[\sqrt{\frac{-3(\vartheta-c^2)}{4\gamma}}\left(\tan\left(\sqrt{\frac{(\vartheta-c^2)}{c^2}}\xi\right) + \cot\left(\sqrt{\frac{(\vartheta-c^2)}{c^2}}\xi\right)\right)\right]^{\frac{1}{2}},$$

(3.17)

where

$$\xi = x \mp \sqrt{\frac{-4\gamma}{3}}a_1 t.$$

(III) If $b = 0$, then we get the same solution (3.8).

4. Other Explicit Results for Equation (3.1):

Here, we utilize a direct approach, employing the Lienard equation, to solve Eq. (3.2). Consequently, we acquire alternative explicit solutions for Eq. (3.1) that diverge from the findings obtained in section 3.2. In order to achieve this objective, we rephrase Eq. (3.3) in the following manner:

$$u'' + \frac{(\vartheta-c^2)}{c^2}u + \frac{\beta}{c^2}u^3 + \frac{\gamma}{c^2}u^5 = 0, \qquad (4.1)$$

if we set:

$$\mathrm{e}_1 = \frac{\vartheta-c^2}{c^2},\quad e_3 = \frac{\beta}{c^2},\quad e_5 = \frac{\gamma}{c^2},$$

in Eq. (4.1) then we obtain the nonlinear Lienard [10]:

$$u''(\xi) + e_1 u(\xi) + e_3 u^3(\xi) + e_5 u^5(\xi) = 0. \qquad (4.2)$$

Eq. (4.2) has multiple solutions see [10]. By utilizing these solutions, we can obtain the following solitary explicit solutions of Eq. (3.1):

Case 1.

$$u(\xi) = \pm\left[\frac{4(c^2-\vartheta)}{c^2\left(\frac{\beta}{c^2}+\frac{\sqrt{3}}{3}\sqrt{\frac{3\beta^2-16\gamma\vartheta+16\gamma c^2}{c^4}}\cosh\left(2\xi\sqrt{\frac{-(\vartheta-c^2)}{c^2}}\right)\right)}\right]^{\frac{1}{2}} \tag{4.3}$$

provided that

$$3\beta^2-16\gamma\vartheta+16\gamma c^2>0\ ,\vartheta-c^2<0\ ,c>0.$$

Case 2.

$$u(\xi) = \pm\left[\frac{4(c^2-\vartheta)}{c^2\left(\frac{\beta}{c^2}+\frac{\sqrt{3}}{3}\sqrt{\frac{16\gamma\vartheta-16\gamma c^2-3\beta^2}{c^4}}\sinh\left(2\xi\sqrt{\frac{-(\vartheta-c^2)}{c^2}}\right)\right)}\right]^{\frac{1}{2}}, \tag{4.4}$$

provided that

$$3\beta^2-16\gamma\vartheta+16\gamma c^2<0,\vartheta-c^2<0.$$

Case 3.

$$u(\xi)=\pm\left[\frac{2(c^2-\vartheta)}{\beta}\left(1+\tanh\left(\xi\sqrt{\frac{-(\vartheta-c^2)}{c^2}}\right)\right)\right]^{\frac{1}{2}}, \tag{4.5}$$

and

$$u(\xi)=\pm\left[\frac{2(c^2-\vartheta)}{\beta}\left(1+\coth\left(\xi\sqrt{\frac{-(\vartheta-c^2)}{c^2}}\right)\right)\right]^{\frac{1}{2}}, \tag{4.6}$$

provided that

$$\frac{3\beta^2-16\gamma\vartheta+16\gamma c^2}{c^4}=0,\qquad \vartheta-c^2<0,\gamma<0,$$

$$\beta>0.$$

Case 4.

$$u(\xi) = \pm\left[\frac{4(c^2-\vartheta)}{c^2\left(\frac{\beta}{c^2}+\frac{\sqrt{3}}{3}\sqrt{\frac{3\beta^2-16\gamma\vartheta+16\gamma c^2}{c^4}}\cos\left(2\xi\sqrt{\frac{(\vartheta-c^2)}{c^2}}\right)\right)}\right]^{\frac{1}{2}} \tag{4.7}$$

and

$$u(\xi) = \pm\left[\frac{4(c^2-\vartheta)}{c^2\left(\frac{\beta}{c^2}+\frac{\sqrt{3}}{3}\sqrt{\frac{3\beta^2-16\gamma\vartheta+16\gamma c^2}{c^4}}\sin\left(2\xi\sqrt{\frac{(\vartheta-c^2)}{c^2}}\right)\right)}\right]^{\frac{1}{2}} \tag{4.8}$$

provided that

$$3\beta^2-16\gamma\vartheta+16\gamma c^2>0,\qquad \vartheta-c^2>0.$$

We now derive additional solutions for Eq. (3.1) using Jacobi-elliptic functions:

Case 5.

$$u(\xi)=\pm\left[\frac{-3\beta}{8\gamma}\left(1+\mathrm{sn}\left(\frac{\sqrt{3}\beta}{4rc\sqrt{-\gamma}}\xi,r\right)\right)\right]^{\frac{1}{2}}, \tag{4.9}$$

provided that

$$\vartheta=\frac{3\beta^2(5r^2-1)+64\gamma r^2c^2}{64\gamma r^2},\qquad \beta>0,\qquad \gamma<0.$$

If $r\to 1$, then we have the dark-soliton wave solution:

$$u(\xi) = \pm\left[\frac{-3\beta}{8\gamma}\left(1+\tanh\left(\frac{\sqrt{3}\beta}{4c\sqrt{-\gamma}}\xi\right)\right)\right]^{\frac{1}{2}}, \qquad (4.10)$$

Case 6.

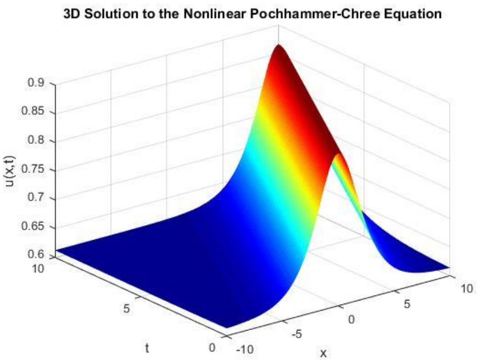
$$u(\xi) = \pm\left[\frac{-3\beta}{8\gamma}\left(1+\mathrm{cn}\left(\frac{\sqrt{3}\beta}{4rc\sqrt{\gamma}}\xi, r\right)\right)\right]^{\frac{1}{2}}, \qquad (4.11)$$

provided that

$$\vartheta = \frac{3\beta^2(4r^2+1)+64\gamma r^2c^2}{64\gamma r^2}, \qquad \beta < 0, \qquad \gamma > 0.$$

If $r \to 1$, then we have the bright-soliton wave solution:

$$u(\xi) = \pm\left[\frac{-3\beta}{8\gamma}\left(1+\mathrm{sech}\left(\frac{\sqrt{3}\beta}{4c\sqrt{\gamma}}\xi\right)\right)\right]^{\frac{1}{2}}, \qquad (4.12)$$



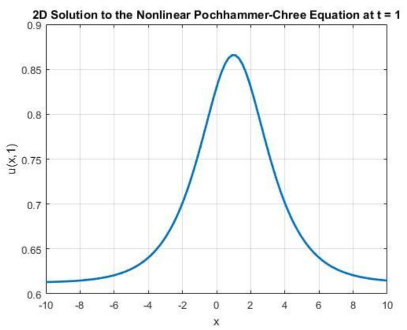


Figure 2. The profile of the bright-soliton solution (4.12) with $c = 1,\ \beta = -2, \gamma = 2$.

Case 7.

$$u(\xi) = \pm\left[\frac{-3\beta}{8\gamma}\left(1+\mathrm{dn}\left(\frac{\sqrt{3}\beta}{4c\sqrt{\gamma}}\xi, r\right)\right)\right]^{\frac{1}{2}}, \qquad (4.13)$$

provided that

$$\vartheta = \frac{3\beta^2(r^2+4)+64\gamma c^2}{64\gamma}, \qquad \beta < 0, \qquad \gamma > 0.$$

If $r \to 1$, then we obtain as the same bright-soliton wave solution (4.12).

5. Conclusions

The tanh-function expansion scheme and a direct approach based on the Liénard equation are used in this article to obtain many new explicit wave solutions to the nonlinear Pochhammer-Chree equation. Comparing our new results obtained in this paper with the well-known results in [17], we conclude that all results obtained in article are new and not found elsewhere. 2D and 3D graphs of certain selected solutions were depicted to show the physical structure of different solutions types. The scheme employed in this article is effective and can be applied to other nonlinear models in the field of mathematical physics. Furthermore, with the aid of Maple software, we have demonstrated that all the solutions obtained in this paper satisfy the original governing equations.

References

[1] Zayed, E. M., Alurrfi, K. A., (2015), On solving the nonlinear Biswas-Milovic equation with dual-power law nonlinearity using the extended tanh-function method. J. Adv. Phys., 11, 3002-3012.

[2] Zayed, E. M. E., Alurrfi, K. A. E., (2013), The modified extended tanh-function method and its applications to the

generalized KdV-mKdV equation with any-order nonlinear terms. Int. J. Environ. Eng. Sci. Technol. Res., 8, 165-170.

[3] Rabie, W. B., Ahmed, H. M., Darwish, A., Hussein, H. H., (2023), Construction of new solitons and other wave solutions for a concatenation model using modified extended tanh-function method. Alex. Eng. J., 74, 445-451.

[4] Ananna, S. N., Gharami, P. P., An, T., Asaduzzaman, M., (2022), The improved modified extended tanh-function method to develop the exact travelling wave solutions of a family of 3D fractional WBBM equations. Results Phys., 41, 105969.

[5] El-Wakil, S. A., Abdou, M. A., (2007), Modified extended tanh-function method for solving nonlinear partial differential equations. Chaos Solitons & Fractals, 31, 1256-1264.

[6] Shahoot, A. M., Alurrfi, K. A., Elmrid, M. O., Almsiri, A. M., Arwiniya, A. M., (2019), The (G′/G)-expansion method for solving a nonlinear PDE describing the nonlinear low-pass electrical lines. J. Taibah Univ. Sci., 13, 63-70.

[7] Zayed, E. M., Alurrfi, K. A., Arnous, A. H., Hashemi, M. S., Bayram, M., (2024), Effects of high dispersion and generalized non-local laws on optical soliton perturbations in magneto-optic waveguides with sextic-power law refractive index. Nonlinear Dyn., 112, 8507-8525.

[8] Zhang, J., Wei, X., Lu, Y., (2008), A generalized (G′/G)-expansion method and its applications. Phys. Lett. A, 372, 3653-3658.

[9] Wang, M., Li, X., Zhang, J., (2008), The (G′/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. Phys. Lett. A, 372, 417-423.

[10] Alurrfi, K. A., Shahoot, A. M., Elhasadi, O. I., (2024), Exact solutions for the GKdV–mKdV equation with higher-order nonlinear terms using the generalized (G′/G,1/G)-expansion method and the generalized Liénard equation. Ric. Mat., 73, 887–905.

[11] He, J. H., Wu, X. H., (2006), Exp-function method for nonlinear wave equations. Chaos, Solitons Fractals, 30, 700-708.

[12] Zayed, E. M. E., Alurrfi, K. A. E., (2015), A new Jacobi elliptic function expansion method for solving a nonlinear PDE describing pulse narrowing nonlinear transmission lines. J. Part. Diff. Eq., 28, 128-138.

[13] Du, Y., Yin, T., Pang, J., (2024), The exact solutions of Schrödinger–Hirota equation based on the auxiliary equation method. Opt. Quantum Electron., 56, 712.

[14] Zayed, E. M. E, Alurrfi, K. A. E., (2015), The generalized projective Riccati equations method and its applications for solving two nonlinear PDEs describing microtubules, Int. J. Phys. Sci., 13, 391-402.

[15] Arnous, A. H., Hashemi, M. S., Nisar, K. S., Shakeel, M., Ahmad, J., Ahmad, I., Shah, N. A., (2024), Investigating solitary

wave solutions with enhanced algebraic method for new extended Sakovich equations in fluid dynamics. Results in Phys., 57, 107369.

[16] Zayed, E. M., Alurrfi, K. A., Alshbear, R. A., (2023), On application of the new mapping method to magneto-optic waveguides having Kudryashov's law of refractive index. Optik, 287, 171072.

[17] Zayed, E. M., Alurrfi, K. A., (2014), The $(G'/G,1/G)$-expansion method and its applications for solving two higher order nonlinear evolution equations, Math. Probl. Eng., Volume 2014, Article ID 746538, 20 pages.