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بعض خصائص تفاضل التبعية لفصول التبعية باستخدام مؤثر تفاضلي معمم

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Some properties of Differential Subordination for the Subordination Class with the Generalized Derivative Operator

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الملخص:

في ورقة البحث هده استخدمنا المؤثر التفاضلي المعمم $I^m(\lambda_1,\lambda_2,\ell,n)$ الذي عرفناه باستخدام الضرب الديكارتي Hadamard product في قرص الوحدة $\mathbb{U}$ ،وتم إدخال تفاضل التبعية في الفصول الجديدة المعرفة بواسطه هذا المؤثر التفاضلي المعمم وأيضا تم تحقيق علاقات خاصه لتفاضل التبعية لهدا المؤثر باستخدام خصائص مفهوم التبعية .

الكلمات الدالة: الدوال المحدبة، الضرب الديكارتي، المؤثر التفاضلي، الدوال النجمية، تفاضل التبعية.

Abstract

In this paper ,we will using a generalized derivative operator in the unit disk $\mathbb{U} = \{z: |z| < 1 ; z \in \mathbb{C}\}$ which is defined by Hadamard product , and introduced differential Subordination for a new class define by this operator, and satisfy its specific relationship to derive the subordination for this operator by using properties of subordination concept .

Keywords: Convex functions, Cartesian multiplication, differential operator, stellar functions, dependency differential.

1. Introduction:

The theory of univalent functions is one of the most important subjects in geometric function theory. The study of univalent functions was initiated by Koebe(Koebe, 1909) in 1907. One of the major problems in this field had been the Bieberbach (Bieberbach, 1916) conjecture dating from the year 1916, which asserts that the modulus of the nth Taylor coefficient of each normalized analytic univalent function is bounded by n. The conjecture was not completely solved until 1984 by French American mathematician Louis de Branges (De Branges, 1985).

Now, let $\mathcal{A}$ denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \mathbb{U}), \qquad (1)$$

which is normalized power series , where a_k is a complex number and $f(z)$ is functions in the open unit disk $\mathbb{U} = \{z: |z| < 1; z \in \mathbb{C}\}$.

The class $\mathcal{S}$ of univalent functions in $\mathcal{A}$ normalized with the conditions $f(0) = f'(0) - 1 = 0$.

Now , Let $\mathcal{S}^*, C$ be the subclasses of $\mathcal{A}$ then we say that f is a starlike function if :

$$\left\{f \in \mathcal{S}^* : \mathrm{Re}\left\{\frac{z f'(z)}{f(z)}\right\} > 0\right\}, z \in \mathbb{U}.$$

And f is convex function if :

$$\left\{f \in C : \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0\right\}, z \in \mathbb{U}.$$

If the function f and g are analytic in $\mathbb{U}$,then we say f is subordinate to g in $\mathbb{U}$,written as $f \prec g$ if there is a Schwarz function $v(z)$ analytic in $\mathbb{U}$, with $|v(z)| < 1$, So that $f(z) = g(v(z))$; $z \in \mathbb{U}$.

If the function g is univalent in $\mathbb{U}$ then the subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) = g(\mathbb{U})$.

The Hadamard product of two analytic functions f and g denoted by $f * g$, where $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ *and* $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$; $(z \in \mathbb{U})$, is defined by

$$(f*g)(z)=f(z)*g(z)=z+\sum_{k=2}^{\infty}a_k b_k z^k \quad , \quad (z\in\mathbb{U}).$$

And by using this product, Amer and Darus (Amer & Darus, 2011) they have recently introduced a new generalized derivative operator.

Definition 1:

For $f\in\mathcal{A}$ the operator $I^m(\lambda_1,\lambda_{2,}\ell,n)$ is defined by $I^m(\lambda_1,\lambda_{2,}\ell,n):\mathcal{A}\longrightarrow\mathcal{A}$.

$$I^m(\lambda_1,\lambda_{2,}\ell,n)f(z)=\varphi^m(\lambda_1,\lambda_2,\ell)(z)*R^nf(z) \quad (z\in\mathbb{U}) \quad , \qquad \rightarrow \ (2)$$

where $\mathrm{m}\in\mathrm{N}_0=\{0,1,2,\ldots\ldots\}$ and $\lambda_2\geq\lambda_1\geq 0,\ \ell\geq 0$. and $R^nf(z)$ denotes the Ruseheweyh derivative operator and given by

$$R^nf(z)=z+\sum_{k=2}^{\infty}c(n,k)\ a_k b_k z^k \ , \ (n\in N_0, z\in\mathbb{U}),$$

where $c(n,k)=\frac{(n+1)_{k-1}}{(1)_{k-1}}$.

If $f(z)$ given by (1), then we easily find from the equality (2) that

$$I^m(\lambda_1,\lambda_{2,}\ell,n)f(z)=z+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+\ell)^{m-1}}{(1+\ell)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)a_k z^k,$$

where $n,m\in N_0=\{0,1,2,\ldots\ldots\}$ and $\lambda_2\geq\lambda_1\geq 0,\ \ell\geq 0.$

Some special cases of this operator includes:

- The Ruscheweyh derivative operator (Ruscheweyh, 1975) in the cases :

$$I^1(\lambda_1,0,l,n)\equiv I^1(\lambda_1,0,0,n)\equiv I^1(0,0,l,n)\equiv I^0(0,\lambda_2,0,n)\equiv I^0(0,0,0,n)\equiv I^{m+1}(0,0,l,n) \equiv I^{m+1}(0,0,0,n)\equiv\mathbb{R}^n.$$

- The Salagean derivative operator(Salagean, 2006):

$$I^{m+1}(1,0,0,0)\equiv D^n.$$

- The generalized Ruscheweyh derivative operator(Shaqsi & Darus, 2008) :

$$I^2(\lambda_1,0,0,n)\equiv R_\lambda^n.$$

- The generalized Salagean derivative operator introduced by Al–Oboudi (Al–Oboudi, 2004) :

$$I^{m+1}(\lambda_1,0,0,0)\equiv D_\beta^n.$$

Using simple computation one obtains the next result

$$(\ell+1)I^{m+1}(\lambda_1,\lambda_{2,}\ell,n)f(z) =$$

$$(1+\ell-\lambda_1)[I^m(\lambda_1,\lambda_{2,}\ell,n) * \varphi^1(\lambda_1,\lambda_2,\ell)(z)]f(z) + \lambda_1 z\left[I^m(\lambda_1,\lambda_{2,}\ell,n) * \varphi^1(\lambda_1,\lambda_2,\ell)(z)\right]', \quad \rightarrow \quad (3)$$

where $(z \in \mathbb{U})$ and $\varphi^1(\lambda_1,\lambda_2,\ell)(z)$ analytic function .

Definition 2:

Let $Y: \mathbb{C}^3 \times \mathbb{U} \rightarrow \mathbb{C}$ and $h(z)$ be univalent in $\mathbb{U}$ .If $p(z)$ is analytic in $\mathbb{U}$,that fulfills the second –order differential subordination (Cotîrlă & Juma, 2023):

$$Y(p(z), zp'(z), z^2 p''(z); z) \prec h(z) \quad , \quad \rightarrow \quad (4)$$

then $p(z)$ is the differential subordination Solution of (4).

Definition 3:

Let Q be the collections of functions f that are analytic and injective on $\overline{\mathbb{U}}/E(f)$. when

$$E(f) = \{\varsigma \in \partial\mathbb{U} : lim_{z\rightarrow\varsigma} f(z) = \infty\} \text{ and } f'(z) \neq 0 \ for \ \varsigma \in \partial\mathbb{U}/E(f) \quad .$$

(Cotîrlă & Juma, 2023).

Lemma 1:

Let $p_1(z)$ be the univalent function in $\mathbb{U}$. and let Σ and ϑ be holomorphic in a domain $p_1(\mathbb{U}) \subset D$, with $\vartheta(z) \neq 0$, when $z \in p_1(\mathbb{U})$.

Set $O(z) = z\, p_1(z)\vartheta(p_1(z))$ and $h(z) = \Sigma(p_1(z)) + O(z)$. suppose that :

i) O is starlike in $\mathbb{U}$.

ii) $Re\left(\frac{zh'(z)}{O(z)}\right) > 0$, $z \in \mathbb{U}$.

If $p_2(z)$ is holomorphic in $\mathbb{U}$ with $p_2(0) = p_1(0)$, $p_2(\mathbb{U}) \subset D$,and

$$\Sigma(p_2(z)) + zp_2'(z)\vartheta(p_2(z)) \prec \Sigma(p_1(z)) + zp_1'(z)\vartheta(p_1(z)),$$

then $$p_2(z) \prec p_1(z).$$

(Miller & Mocanu, 2003)

Lemma 2:

Let $p_1(z)$ be convex in $\mathbb{U}$and $\beta_1 \in \mathbb{C}$, $\beta_2 \in \mathbb{C}^*$ with

$$Re\left(1+\frac{p_1''(z)}{p_1'(z)}\right) > max\left\{0\,, -Re\frac{\beta_1}{\beta_2}\right\}.$$

If $p_2(z)$ is holomorphic in $\mathbb{U}$ and $\beta_1 p_2(z) + \beta_2 z p_2'(z) \prec \beta_1 p_1(z) + \beta_2 z p_1'(z),$

then $$p_2(z) \prec p_1(z) \quad .$$

(Shanmugam, Sivasubramanian, & Srivastava, 2006).

Theorem 1:

Let b be convex univalent in $\mathbb{U}$ with $b(0) = 1, a_1 > 0$,$0 \neq a_2 \in \mathbb{C}$ and suppose :

$$Re\left(1+\frac{b''(z)}{b'(z)}\right) > max\left\{0\,, -Re\frac{a_1}{a_2}\right\} .$$

If $f \in \mathcal{A}$ it satisfies the Subordination :

$$\left(1 - a_2\left(\frac{1}{\lambda_1}\right)\right)\left(\frac{I^m(\lambda_1,\lambda_2,\ell,n)f(z)}{z}\right)^{a_1} + a_2\left(\frac{1}{\lambda_1}\right)\left(\frac{I^{m+1}(\lambda_1,\lambda_2,\ell,n)f(z)}{I^m(\lambda_1,\lambda_2,\ell,n)f(z)}\right)^{a_1} \left(I^m(\lambda_1,\lambda_2,\ell,n)f(z)\right)^{a_1}$$

$$\prec b(z) + \frac{a_2}{a_1} z\ b'(z),$$

then

$$\left(\frac{I^m(\lambda_1,\lambda_2,\ell,n)f(z)}{z}\right)^{a_1} \prec b(z)\ .$$

Proof:

Consider

$$q(z) = \left(\frac{I^m(\lambda_1,\lambda_2,\ell,n)f(z)}{z}\right)^{a_1}. \qquad \rightarrow (5)$$

Then

$$q'(z) = a_1\left(\frac{I^m(\lambda_1,\lambda_2,\ell,n)f(z)}{z}\right)^{a_1-1}\left(\frac{z[I^m(\lambda_1,\lambda_2,\ell,n)f(z)]' - I^m(\lambda_1,\lambda_2,\ell,n)f(z)}{z^2}\right)$$

$$q'(z) = a_1\left(\frac{I^m(\lambda_1,\lambda_2,\ell,n)f(z)}{z}\right)^{a_1}\left(\frac{z}{I^m(\lambda_1,\lambda_2,\ell,n)f(z)}\right)$$
$$\left(\frac{z[I^m(\lambda_1,\lambda_2,\ell,n)f(z)]' - I^m(\lambda_1,\lambda_2,\ell,n)f(z)}{z^2}\right)$$

$$\Rightarrow q'(z) = a_1 \left(\frac{I^m(\lambda_1,\lambda_2,\ell,n)f(z)}{z}\right)^{a_1} \left(\frac{[I^m(\lambda_1,\lambda_2,\ell,n)f(z)]'}{I^m(\lambda_1,\lambda_2,\ell,n)f(z)} - \frac{1}{z}\right).$$

From (5) ,we have

$$z\frac{q'(z)}{q(z)} = a_1 z \left(\frac{[I^m(\lambda_1,\lambda_2,\ell,n)f(z)]'}{I^m(\lambda_1,\lambda_2,\ell,n)f(z)} - \frac{1}{z}\right)$$
$$= a_1\left(\frac{z[I^m(\lambda_1,\lambda_2,\ell,n)f(z)]'}{I^m(\lambda_1,\lambda_2,\ell,n)f(z)} - 1\right).$$

By using (3) , we obtain

$$z\frac{q'(z)}{q(z)} = a_1 z \left(\frac{(\ell+1)I^{m+1}(\lambda_1,\lambda_2,\ell,n)f(z)}{\lambda_1 z\left(\varphi^1(\lambda_1,\lambda_2,\ell)(z)\right)' I^m(\lambda_1,\lambda_2,\ell,n)f(z)} - \frac{(1+\ell-\lambda_1)\varphi^1(\lambda_1,\lambda_2,\ell)(z)}{\lambda_1 z\left(\varphi^1(\lambda_1,\lambda_2,\ell)(z)\right)'} - 1\right),$$

and

$$\frac{a_2}{a_1} zq'(z) = \frac{a_2}{\lambda_1}\left(\frac{(\ell+1)I^{m+1}(\lambda_1,\lambda_2,\ell,n)f(z)}{\left(\varphi^1(\lambda_1,\lambda_2,\ell)(z)\right)' I^m(\lambda_1,\lambda_2,\ell,n)f(z)}\right).\left(\frac{I^m(\lambda_1,\lambda_2,\ell,n)f(z)}{z}\right)^{a_1}$$
$$-\frac{a_2}{\lambda_1}\frac{(1+\ell-\lambda_1)\varphi^1(\lambda_1,\lambda_2,\ell)(z)}{\left(\varphi^1(\lambda_1,\lambda_2,\ell)(z)\right)'}.\left(\frac{I^m(\lambda_1,\lambda_2,\ell,n)f(z)}{z}\right)^{a_1}$$
$$-a_2\left(\frac{I^m(\lambda_1,\lambda_2,\ell,n)f(z)}{z}\right)^{a_1},$$

$$\Rightarrow \frac{a_2}{a_1} zq'(z) = \frac{a_2}{\lambda_1}\left(\frac{(\ell+1)I^{m+1}(\lambda_1,\lambda_2,\ell,n)f(z)}{\left(\varphi^1(\lambda_1,\lambda_2,\ell)(z)\right)' I^m(\lambda_1,\lambda_2,\ell,n)f(z)}\right).\left(\frac{I^m(\lambda_1,\lambda_2,\ell,n)f(z)}{z}\right)^{a_1}$$
$$-\frac{a_2}{\lambda_1}\left(\frac{I^m(\lambda_1,\lambda_2,\ell,n)f(z)}{z}\right)^{a_1}\left(\frac{(1+\ell-\lambda_1)\varphi^1(\lambda_1,\lambda_2,\ell)(z)}{\left(\varphi^1(\lambda_1,\lambda_2,\ell)(z)\right)'} - \lambda_1\right).$$

By using hypothesis, we obtain

$$b(z) + \frac{a_2}{a_1} zq'(z) \prec b(z) + \frac{a_2}{a_1} zb'(z).$$

Additionally ,apply (Lemma 2), when $\beta_1 = 1$ *and* $\beta_2 = \frac{a_2}{a_1}$ *then*

$$q(z) \prec b(z)$$
$$\therefore \quad \left(\frac{I^m(\lambda_1,\lambda_2,\ell,n)f(z)}{z}\right)^{a_1} \prec b(z).$$

Corollary 1:

Let b be convex univalent in $\mathbb{U}$ with $b(0) = 1, a_1 > 0, 0 \neq a_2 \in \mathbb{C}$ and suppose :

$$Re\left(1+\frac{b''(z)}{b'(z)}\right)>max\left\{0\,,-Re\frac{a_1}{a_2}\right\}.$$

If $f\in\mathcal{A}$ it satisfies the Subordination

$$(1-a_2)\left(\frac{D^{n-1}f(z)}{z}\right)^{a_1}+a_2\left(\frac{D^nf(z)}{(D^{n-1})f(z)}\right)^{a_1}(D^{n-1}f(z))^{a_1}\prec b(z)+\frac{a_2}{a_1}z\,b'(z),$$

then

$$\left(\frac{D^{n-1}f(z)}{z}\right)^{a_1}\prec b(z).$$

(Cotîrlă & Juma, 2023)

Theorem 2:

Let b be convex univalent in $\mathbb{U}, b(0)=1$, and $b(z)\neq 0$ for all $z\in\mathbb{U}$, and suppose that b satisfies :

$$Re\left\{p+\frac{zt\sigma}{za_2}+\frac{z\varepsilon(\sigma+1)}{za_2}(z)+(\sigma-1)\frac{zb'(z)}{b(z)}+\frac{z\,b''(z)}{b'(z)}\right\}>0\,,$$

where $\sigma,\varepsilon,t\in\mathbb{C}$, $a_1>0$,$0\neq a_2\in\mathbb{C}$ *and* $z\in\mathbb{U}$.

Suppose that $z(b(z))^{\sigma-1}\,b'(z)$ is a starlike univalent in $\mathbb{U}$.

If $f\in\mathcal{A}$ it satisfies the Subordination

$$\mathcal{M}(1,m,\lambda_1,\varepsilon,a_1,a_2;z)\prec\big(t+\varepsilon\,b(z)\big)\big(b(z)\big)^{\sigma}+a_2\big(b(z)\big)^{\sigma-1}b'(z)\,,\longrightarrow(6)$$

where

$$\begin{aligned}\mathcal{M}(1,m,\lambda_1,\varepsilon,a_1,a_2;z)&\\ =t&\left(\frac{(\frac{1}{\lambda_1})(I^{m+1}(\lambda_1,\lambda_{2,}\ell,n)f(z))+(1-\frac{1}{\lambda_1})(I^m(\lambda_1,\lambda_{2,}\ell,n)f(z))}{z}\right)^{a_1\sigma}\\ +\varepsilon&\left(\frac{(\frac{1}{\lambda_1})(I^{m+1}(\lambda_1,\lambda_{2,}\ell,n)f(z))+(1-\frac{1}{\lambda_1})(I^m(\lambda_1,\lambda_{2,}\ell,n)f(z))}{z}\right)^{a_1(\sigma+1)}\\ +a_1a_2&\left(\frac{(\frac{1}{\lambda_1})(I^{m+1}(\lambda_1,\lambda_{2,}\ell,n)f(z))+(1-\frac{1}{\lambda_1})(I^m(\lambda_1,\lambda_{2,}\ell,n)f(z))}{z}\right)^{a_1\sigma}.\\ &\left(\frac{(\frac{z}{\lambda_1})(I^{m+1}(\lambda_1,\lambda_{2,}\ell,n)f(z))'+(1-\frac{1}{\lambda_1})(I^m(\lambda_1,\lambda_{2,}\ell,n)f(z))'}{(\frac{1}{\lambda_1})(I^{m+1}(\lambda_1,\lambda_{2,}\ell,n)f(z))+(1-\frac{1}{\lambda_1})(I^m(\lambda_1,\lambda_{2,}\ell,n)f(z))}-p\right)\end{aligned}$$

then

$$\left(\left(\frac{1}{\lambda_1}\right)(I^{m+1}(\lambda_1,\lambda_{2,}\ell,n)f(z)) + \left(1-\frac{1}{\lambda_1}\right)(I^{m}(\lambda_1,\lambda_{2,}\ell,n)f(z)) \right)^{a_1} \prec b(z).$$

Proof :

Let $H(\beta) = (t+\varepsilon\beta)\beta^{\sigma}$ *and* $L(\beta) = a_2(\beta)^{\sigma-1}$; $0 \neq \beta \in \mathbb{C}$. when $H(\beta), L(\beta)$ are analytic in $\mathbb{C}$.

By using (Lemma 1), we obtain

$$G(z) = zb'(z)\, L\,(b(z))$$
$$= a_2 z\,(b(z))^{\sigma-1} b'(z)\ ,$$

and
$$y(z) = H\big(b(z)\big) + G(z)$$
$$= \big(t+\varepsilon b(z)\big)\big(b(z)\big)^{\sigma} + a_2 z\,(b(z))^{\sigma-1} b'(z).$$

Since $z(b(z))^{\sigma-1}\, b'(z)$ is a starlike, then $G(z)$ is starlike in $\mathbb{U}$.

Now
$$y'(z) = \sigma t z\big(b(z)\big)^{\sigma-1} b'(z) + \varepsilon(\sigma+1)\big(b(z)\big)^{\sigma}.zb'(z) + a_2 z\,(b(z))^{\sigma-1} b''(z).z + a_2 z(\sigma-1)\,(b(z))^{\sigma-2} b'(z).z + a_2(b(z))^{\sigma-1} b'(z),$$

and

$$\frac{y'(z)}{G(z)} = \frac{\sigma t}{a_2} + \frac{\varepsilon(\sigma+1)(b(z)}{a_2} + \frac{z\,b''(z)}{b'(z)} + \frac{z(\sigma-1)}{b(z)} + \frac{1}{z}$$

$$\therefore \quad Re\ \left(\frac{y'(z)}{G(z)}\right) = Re\left\{\frac{1}{z} + \frac{\sigma t}{a_2} + \frac{\varepsilon(\sigma+1)(b(z)}{a_2} + \frac{z(\sigma-1)}{b(z)} + \frac{z\,b''(z)}{b'(z)}\right\}.$$

Now, consider

$$q(z) = \left(\frac{(\frac{1}{\lambda_1})(I^{m+1}(\lambda_1,\lambda_{2,}\ell,n)f(z)) + (1-\frac{1}{\lambda_1})(I^{m}(\lambda_1,\lambda_{2,}\ell,n)f(z))}{z} \right)^{a_1} \quad \rightarrow (7)$$

Then

$$q'(z) = a_1 \left(\frac{(\frac{1}{\lambda_1})(I^{m+1}(\lambda_1,\lambda_{2,}\ell,n)f(z)) + (1-\frac{1}{\lambda_1})(I^{m}(\lambda_1,\lambda_{2,}\ell,n)f(z))}{z} \right)^{a_1}.$$
$$\left(\frac{\left(\frac{1}{\lambda_1}\right)(I^{m+1}(\lambda_1,\lambda_{2,}\ell,n)f(z))' + \left(1-\frac{1}{\lambda_1}\right)(I^{m}(\lambda_1,\lambda_{2,}\ell,n)f(z))'}{(\frac{1}{\lambda_1})(I^{m+1}(\lambda_1,\lambda_{2,}\ell,n)f(z)) + (1-\frac{1}{\lambda_1})(I^{m}(\lambda_1,\lambda_{2,}\ell,n)f(z))} - \frac{1}{z} \right).$$

From (7), we obtain

$$t\left(\frac{(\frac{1}{\lambda_1})(I^{m+1}(\lambda_1,\lambda_{2,}\ell,n)f(z))+(1-\frac{1}{\lambda_1})(I^{m}(\lambda_1,\lambda_{2,}\ell,n)f(z))}{z}\right)^{a_1\sigma}$$

$$+\varepsilon\left(\frac{(\frac{1}{\lambda_1})(I^{m+1}(\lambda_1,\lambda_{2,}\ell,n)f(z))+(1-\frac{1}{\lambda_1})(I^{m}(\lambda_1,\lambda_{2,}\ell,n)f(z))}{z}\right)^{a_1(\sigma+1)}$$

$$=t\left(q(z)\right)^{\sigma}+\varepsilon\left(\left(q(z)\right)^{\sigma}q(z)\right)$$

$$=\left(t+\varepsilon\, q(z)\right)\left(q(z)\right)^{\sigma}\quad .$$

and

$$a_1\left(\frac{(\frac{z}{\lambda_1})(I^{m+1}(\lambda_1,\lambda_{2,}\ell,n)f(z))'+(1-\frac{1}{\lambda_1})(I^{m}(\lambda_1,\lambda_{2,}\ell,n)f(z))'}{(\frac{1}{\lambda_1})(I^{m+1}(\lambda_1,\lambda_{2,}\ell,n)f(z))+(1-\frac{1}{\lambda_1})(I^{m}(\lambda_1,\lambda_{2,}\ell,n)f(z))}-p\right)=z\frac{q'(z)}{q(z)},$$

hence

$$a_1a_2\left(\frac{(\frac{1}{\lambda_1})(I^{m+1}(\lambda_1,\lambda_{2,}\ell,n)f(z))+(1-\frac{1}{\lambda_1})(I^{m}(\lambda_1,\lambda_{2,}\ell,n)f(z))}{z}\right)^{a_1\sigma}$$

$$\cdot\left(\frac{(\frac{z}{\lambda_1})(I^{m+1}(\lambda_1,\lambda_{2,}\ell,n)f(z))'+(1-\frac{1}{\lambda_1})(I^{m}(\lambda_1,\lambda_{2,}\ell,n)f(z))'}{(\frac{1}{\lambda_1})(I^{m+1}(\lambda_1,\lambda_{2,}\ell,n)f(z))+(1-\frac{1}{\lambda_1})(I^{m}(\lambda_1,\lambda_{2,}\ell,n)f(z))}-p\right)=a_2\left(q(z)\right)^{\sigma}z\frac{q'(z)}{q(z)}$$

$$=a_2\, z\left(q(z)\right)^{\sigma-1}q'(z)\,.$$

$$\therefore\mathcal{M}(1,m,\lambda_1,\varepsilon,a_1,a_2;z)=\left(t+\varepsilon\, q(z)\right)\left(q(z)\right)^{\sigma}+a_2\, z\left(q(z)\right)^{\sigma-1}q'(z).$$

From (6) we obtain

$$\left(t+\varepsilon\, q(z)\right)\left(q(z)\right)^{\sigma}+a_2\, z\left(q(z)\right)^{\sigma-1}q'(z)\prec\left(t+\varepsilon\, b(z)\right)\left(b(z)\right)^{\sigma}+a_2\left(b(z)\right)^{\sigma-1}b'(z),$$

and by using (Lemma 1), we obtain

$$\mathrm{q(z)}\prec\mathrm{b(z)}\ .$$

$$\therefore\left(\frac{(\frac{1}{\lambda_1})(I^{m+1}(\lambda_1,\lambda_{2,}\ell,n)f(z))+(1-\frac{1}{\lambda_1})(I^{m}(\lambda_1,\lambda_{2,}\ell,n)f(z))}{z}\right)^{a_1}\prec b(z).$$

Corollary 2:

Let b be convex univalent in $\mathbb{U}$,$b(0)=1$, and $b(z)\neq 0$ for all $z\in\mathbb{U}$,and Suppose that b satisfies :

$$Re\left\{p+\frac{t\sigma}{a_2}+\frac{\varepsilon(\sigma+1)}{a_2}(z)+(\sigma-1)\frac{zb'(z)}{b(z)}+\frac{z\,b''(z)}{b'(z)}\right\}>0\,,$$

where $\sigma,\varepsilon,t\in\mathbb{C}$, $a_1>0\,, 0\neq a_2\in\mathbb{C}$ and $z\in\mathbb{U}$.

Suppose that $z(b(z))^{\sigma-1}\, b'(z)$ is a starlike univalent in $\mathbb{U}$.

If $f \in \mathcal{A}$ it satisfies the Subordination

$$\mathcal{M}(\sigma, t, \varepsilon, h, \mu, a_1, a_2; z) \prec \left(t + \varepsilon\, b(z)\right)\left(b(z)\right)^{\sigma} + a_2\left(b(z)\right)^{\sigma-1} b'(z),$$

then

$$\therefore \left(\frac{D^n f(z)}{z}\right)^{a_1} \prec\; b(z).$$

(Cotîrlă & Juma, 2023)

Many other work on analytic functions functions related to derivative operator and integral operator can be read in (Shmella & Amer, 2024) (Alabbar, Darus, & Amer, 2023), (Amer & Alabbar, 2017) and(Amer, 2016) .

2. Conclusion:

In this work , we defined generalized derivative operator using Hadamard product in the unit disk $\mathbb{U} = \{z: |z| < 1; z \in \mathbb{C}\}$. and defined the Subordination class by this operator, and satisfied its specific relationship to derive the subordination for this operator using the properties of subordination concept.

References

Al-Oboudi, F. M. (2004). On univalent functions defined by a generalized Sălăgean operator. *International Journal of Mathematics and Mathematical Sciences, 2004*, 1429-1436.

Alabbar, N., Darus, M., & Amer, A. (2023). Coefficient Inequality and Coefficient Bounds for a New Subclass of Bazilevic Functions. *Journal of Humanitarian and Applied Sciences, 8*(16), 496-506.

Amer, A. A. (2016). *Second Hankel Determinant for New Subclass Defined by a Linear Operator.* Paper presented at the Computational Analysis: AMAT, Ankara, May 2015 Selected Contributions.

Amer, A. A., & Alabbar, N. M. (2017). Properties of Generalized Derivative Operator to A Certain Subclass of Analytic Functions with Negative Coefficients.

Amer, A. A., & Darus, M. (2011). On some properties for new generalized derivative operator. *Jordan Journal of Mathematics and Statistics (JJMS), 4*(2), 91-101.

Bieberbach, L. (1916). Uber die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln. *Sitzungsberichte Preussische Akademie der Wissenschaften, 138*, 940-955.

Cotîrlă, L.-I., & Juma, A. R. S. (2023). Properties of differential subordination and superordination for multivalent functions associated with the convolution operators. *Axioms, 12*(2), 169.

De Branges, L. (1985). A proof of the Bieberbach conjecture. *Acta Mathematica, 154*(1), 137-152.

Koebe, P. (1909). Ueber die Uniformisierung beliebiger analytischer Kurven.(Vierte Mitteilung). *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, 1909*, 324-362.

Miller, S. S., & Mocanu, P. T. (2003). Subordinants of differential superordinations. *Complex variables, 48*(10), 815-826.

Ruscheweyh, S. (1975). New criteria for univalent functions. *Proceedings of the American Mathematical Society, 49*(1), 109-115.

Salagean, G. S. (2006). *Subclasses of univalent functions.* Paper presented at the Complex Analysis—Fifth Romanian-Finnish Seminar: Part 1 Proceedings of the Seminar held in Bucharest, June 28–July 3, 1981.

Shanmugam, T., Sivasubramanian, S., & Srivastava, H. (2006). Differential sandwich theorems for certain subclasses of analytic functions involving multiplier transformations. *Integral Transforms and Special Functions, 17*(12), 889-899.

Shaqsi, K., & Darus, M. (2008). An operator defined by convolution involving the polylogarithms functions. *Journal of Mathematics and Statistics, 4*(1), 46.

Shmella, E. K., & Amer, A. A. (2024). Estimation of the Bounds of Univalent Functional of Coefficients Apply the Subordination Method.