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أ. إيمان عمران نصر

قسم الرياضيات، كلية التربية، جامعة خليج السدرة، ليبيا

imannassr@gmail.com

Introduction to the Theory and Application of the Laplace Transformation

Iman Omran Nassr

Department of Mathematics, Faculty of Education Gulf of Sidra University, Libya

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الملخص:

يعد تحويل لابلاس أداة مهمة لحل مشكلات المعادلات التفاضلية العادية الخطية ومسائل القيمة الأولية ونظام المعادلات التفاضلية العادية الخطي. يحتوي هذا البحث على تعريف تحويل لابلاس وبعض خصائصه وذكر براهينه. كما أنه يحتوي بعض الأمثلة لشرح استخدام تحويل لابلاس، وأيضا، فإنه يتضمن جدول تحويلات لابلاس لبعض الدوال.

الكلمات الدالة: تحويل لابلاس، معكوس تحويل لابلاس، خصائص، تكامل، المعادلات.

Abstract

Laplace Transform is a significant device for solving linear ODE's and initial value problems and a system of linear ODEs. This paper contains the definition of the Laplace Transform and some of its properties and its proofs are quoted. Also, it has some examples to explain the use of Laplace Transform. Moreover, it involves a table of Laplace Transformation of some functions.

Keywords: Laplace transform, inverse Laplace transform, properties, integration, equations.

1-Introduction

Laplace Transform is a mathematical topic. It represents a useful tool in many fields. Laplace Transform is an integral transform because it shifts a function into an alternative function by applying integration techniques. Studying a function $f(t)$, $L\{f(t)\}$ will denote its Laplace Transform, wherever L is the Laplace factor works on the time domain function *f(t)*. Laplace Transform is employed in solving differential and integral equations.

2− Laplace Transform

2.1− Laplace Transform definition

Given a function *f(t)* where t≥ 0, its Laplace Transform is *F(s)*, and is symbolized by *L{f}*

i.e.

$$F(s) = L\{f\} = \int_0^\infty e^{-st} f(t)dt \qquad (1)$$

Note that we should suppose that *f(t)* satisfies some condition so the integral above exists, On the other hand, we call Laplace Transform an integral transform.

Notation: 2.2−

− The original function depends on *t* and the new function *F* depends on *s*.

2.3−Piecewise continuous function:

A function *f(t)* is stated piecewise continuous on any interval $a \leq t \leq b$ if *f(t)* is defined on $[a, b]$, and if *f(t)* continuous of a limited number of subintervals split from that interval.

2.4− exponential order Definition:

A function *f(t)* is stated to be in exponential order if $\lim_{t\to\infty} e^{-st} f(t) = 0$.

2.5−Existence of Laplace Transform:

Laplace transform of *f(t)* is said to exist if *f(t)* is defined and continuous for all $t \geq 0$ and a piecewise continuous and an exponential order.

2.6− inverse of Laplace Transform

the inverse transform of *F(s)* is the function *f(t)* presented in (1) and is symbolized by $L^{-1}\{F(s)\}$; hence

$$F(s) = l\{f(t)\} \;\leftrightarrow\; f(t) = L^{-1}\{F(s)\} \qquad (2)$$

From (1) and (2) we get

$$L^{-1}\{L(f)\} = f \quad and \quad L\{\, L^{-1}(F)\} = F$$

Example 1: the Laplace Transform of $f(t) = e^{kt}$is given by

$$f(t) = \frac{1}{s-k}$$

proof: from (1) we have

$$L\{f(t)\} = L(e^{kt}) = \int_0^{\infty} e^{-st}\, e^{kt} dt$$

$$= \int_0^{\infty} e^{(k-s)t} dt = \frac{1}{s-k} e^{(k-s)t}\Big|_0^{\infty}$$

$$= \frac{1}{s-k}$$

2.7−Properties of Laplace Transform: will considerate some main Laplace Transform properties

2.7.1−Theorem (linearity): for any function *f(t), g(t)* which its Laplace Transforms exist and for every fixed *a* and *b* the transform of $af(t) + bg(t)$ exists and

$$L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\} \qquad (3)$$

Proof: since integration is a linear procedure by (1) we have

$$L\{af(t) + bg(t)\} = \int_0^{\infty} e^{-st}\, [af(t) + bg(t)]dt$$

$$= a\int_0^{\infty} e^{-st} f(t)dt + b\int_0^{\infty} e^{-st} g(t)dt$$

$$= aL\{f(t) + bL\{g(t)\}\, .$$

Note if we take Laplace Transform inverse we have

$$L^{-1}\{af(t) + bg(t)\} = aL^{-1}\{f(t)\} + bL^{-1}\{g(t)\}$$

Change of scale property:2.72.

A linear division or multiplication of any constant with the variable is recognized as scaling. So, if *L{f(t)} =F(S)*, by scaling property

$$L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right) \qquad (4)$$

And so if $f(t) = L^{-1}\{F(s)\}$

taking Laplace transform inverse $L^{-1}\left\{\frac{1}{a}F\left(\frac{s}{a}\right)\right\} = f(at)$

2.7.3−Multiplication of power of variable

multiplying the premier function *f(t)* with the powers of the variable *t*. The Laplace Transform is

$$L\{\,t^n f(t)\} = \;(-1)^n \frac{d^n}{ds^n} F(s) \qquad (5)$$

2.7.4−First Shifting:

Theorem: if *F(s)* is the transform of *f(t)* (where *s>m* for some *m*) thereafter *F(s−a)* is the transform of $e^{at}f(t)$ (where *s−a>m)*

$$L\{e^{at}f(t)\} = F(s-a) \qquad (6)$$

Or, by taking the inverse of two sides together of (4) we have

$$e^{at}f(t) = L^{-1}\{F(s-a)\}$$

Proof: by definition (1)

$$L\{e^{at}f(t)\} = \int_0^\infty e^{-st}\,(e^{at}f(t))dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)dt = F(s-a)$$

Example 2: the Laplace Transform of *f(t)=cos(wt)* and *f(t)=sin(wt)* is given by

$$L\{\cos(wt)\} = \frac{s}{s^2+w^2} \qquad , \quad L\{\sin(wt)\} = \frac{w}{s^2+w^2}$$

by order write $L_c = L\{\cos(wt)\}$ *and* $L_s = L\{\sin(wt)\}$

Proof: Integrating by parts and observing the following

$$L_c = \int_0^\infty e^{-st}\cos(wt)\,dt = \;\frac{e^{-st}}{-s}\cos(wt)\bigg|_0^\infty - \frac{w}{s}\int_0^\infty e^{-st}\sin(wt)\,dt = \frac{1}{s} - \frac{w}{s}\,L_s$$

$$, L_s = \int_0^\infty e^{-st}\sin(wt)\,dt = \;\frac{e^{-st}}{-s}\sin(wt)\bigg|_0^\infty + \frac{w}{s}\int_0^\infty e^{-st}\cos(wt)\,dt = \frac{w}{s}\,L_c$$

By substituting L_s into the formula for L_c on the right and substituting L_c into the formula for L_s on the right we get

$$L_c = \frac{1}{s} - \frac{w}{s}\left(\frac{w}{s} L_c\right), \qquad L_c\left(1 + \frac{w^2}{s^2}\right) = \frac{1}{s} \ , \qquad L_c = \frac{s}{s^2 + w^2}$$

$$L_s = \frac{w}{s}\left(\frac{1}{s} - \frac{w}{s} L_s\right), \qquad L_s\left(1 + \frac{w^2}{s^2}\right) = \frac{w}{s^2}, \qquad L_s = \frac{w}{s^2 + w^2}$$

Notice that from example 2 and the theorem of first shifting we have

$$L\{e^{at}\cos(wt)\} = \frac{s-a}{(s-a)^2 + w^2} \ ,$$

$$L\{e^{at}\sin(wt)\} = \frac{w}{(s-a)^2 + w^2}$$

2.7.5–Second Shifting:

Theorem: if *F(s)* is the transform of *f(t)*, then the "shifted function"

$$g(t) = f(t-a) = \begin{cases} 0 & if \quad t < a \\ f(t-a) & if \quad t > a \end{cases}$$

Has the transform $L\{G(T)\} = e^{-aS}F(s)$ (7)

Proof: applying definition in (1)

$$\mathrm{L\{g(t)\}} = \int_0^\infty e^{-st} g(t)dt$$

$$= \int_0^a e^{-st} g(t)dt + \int_a^\infty e^{-st} g(t)dt$$

$$= 0 + \int_a^\infty e^{-st} f(t-a)dt$$

Now put *t–a=v* $\rightarrow$ *dt=dv*

$$\therefore \mathrm{L\{g(t)\}} = \int_0^\infty e^{-s(a+v)} f(v)dv = e^{-as}\int_0^\infty e^{-sv} f(v)dv$$

$$= e^{-as}\int_0^\infty e^{-st} f(t)dt$$

$$\therefore \mathrm{L\{g(t)\}} = e^{-as}F(s)$$

Example 3: by using the second shifting theorem prove the following

$$L\{g(t)\} = \begin{cases} 0 & if \quad t < 3 \\ (t-3)^{\,4} & if \quad t > 3 \end{cases} = e^{-3s}\frac{4!}{s^5}$$

Proof: here $a = 3, \;\; f(t-a) = (t-3)^{\,4}$

$$f(t) = t^{\,4}$$

$$L\{f(t)\} = L\{t^{\,4}\} = \frac{4!}{s^5} = F(S)$$

$$\therefore L\{g(t)\} = e^{-as}F(s)$$

$$= e^{-3s}\frac{4!}{s^5}$$

Laplace Transform of Derivatives 2.7.6-

One of the main applications of the Laplace Transform is the solution of linear differential equations with constant coefficients in the existence of boundary and initial conditions, where the ODE is transformed into an algebraic equation using the following theorem.

Theorem: if *f(x)* and its derivatives $f'(t), f^{(2)}(t), \ldots\ldots, f^{(n)}(t)$ exists then:

$$L\{f'\} = sL(f) - f(0) \qquad \rightarrow$$

$$L\{f''\} = s^2L(f) - sf(0) - f'(0)$$

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And so, the n^{th} derivative can be expressed as

$$L\{f^{(n)}\} = s^nL(f) - \cdots - s^2f^{n-3}(0) - sf^{n-2}(0) - f^{n-1}(0) \qquad (8)$$

Proof: we going to prove the formula of the first derivative

$$\text{by definition} \qquad \mathrm{L}\{f'(\mathrm{t})\} = \int_0^\infty e^{-st}\,f(t)dt$$

Applying integration by parts

$$L\{f'(\mathrm{t})\} = [e^{-st}\mathrm{f(t)}]_0^\infty - \int_0^\infty (-s)e^{-st}\,f(t)dt$$

$$= f(0) + \mathrm{s}\int_0^\infty e^{-st}\,f(t)dt$$

$$\therefore L\{f'\} = sL(f) - f(0)$$

Example 4: To solve the initial value problem $y' - y = 1, y = 0$ by

Laplace transform we take Laplace transform for the two sides

$$L\{y'\} - L(y) = L(1)$$

$$sL(y) - y(0) - Y(s) = \frac{1}{s}$$

$$sY(s) - y(0) - Y(s) = \frac{1}{s}$$

$$(s-1)Y(S) - 0 = \frac{1}{s} \quad \text{by using partial fraction}$$

$$Y(s) = \frac{1}{s(s-1)}$$

Taking L^{-1} for both sides we get

$$L^{-1}\{Y(s)\} = -L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$y(t) = -1 + e^t$$

2.7.7–Laplace Transform of Integral

integration of *f(t)* corresponds to the division of *L{f(t)}* by *s.*

Theorem: the integrated time domain function by Laplace Transform is

$$\text{L}\{\int_0^t f(u)\,du\} = \frac{1}{s}F(s), thus \ \int_0^t f(u)\,du = L^{-1}\left\{\frac{1}{s}F(s)\right\} \qquad (9)$$

Proof: by definition (1)

$$L\{\int_0^t f(u)\,du\} = \int_0^\infty e^{-st}\,(\int_0^t f(u)\,du)dt$$

Integrating by parts

$$= [\int_0^t f(u)\,du(\frac{-e^{-st}}{s})\Big|_0^\infty - \int_0^\infty \left(\frac{-e^{-st}}{s}\right)dt \int_0^t f(u)\,du$$

but $\frac{d}{dt}\int_0^t f(u)\,du = f(t)$

$$= \int_0^\infty \frac{1}{s}e^{-st}f(t)dt = \frac{1}{s}L\{f(t)\}$$

$$\therefore \; \mathrm{L}\{\int_0^t f(u)\,du\} = \frac{1}{s}F(s)$$

Example 5: Laplace Transform of this integral

$$\int_0^t (\cos(5u) - u^2 + e^{-3u})du \;\; is \;\; \frac{1}{s^2+25} - \frac{2}{s^4} + \frac{1}{s(s+3)}$$

Proof: by definition (1)

$$L\{f(u)\} = L\{\cos(5u) - u^2 + e^{-3u}\} = \frac{s}{s^2+25} - \frac{2}{s^3} + \frac{1}{s+3} = F(s)$$

And by the theorem of Laplace Transform Integral

$$\mathrm{L}\left\{\int_0^t f(u)\,du\right\} = \mathrm{L}\left\{\int_0^t (\cos(5u) - u^2 + e^{-3u})\,du\right\} = \frac{1}{s}F(s)$$

$$= \frac{1}{s^2+25} - \frac{2}{s^4} + \frac{1}{s(s+3)}$$

2.8- Unit Step Function (Heaviside Function)

The unit step function *u(t–a)* is *0* for *t<a,* has a jump of size *1* at *t=a,* and is *1* for *t>0*, in a formula:

$$u(t-a) = \begin{cases} 0 & if\ t < 0 \\ 1 & if\ t > a \end{cases} \qquad ,(a \geq 0)$$

taking Laplace Transform

$$L\{u(t-a)\} = \int_0^\infty e^{-st}\,u(t-a)dt = \int_0^a e^{-st}\,.\,0\ dt \int_a^\infty e^{-st}\,.\,1dt \;= \left.\frac{-e^{-st}}{s}\right|_{t=a}^{\infty}$$

Notice that we begin the integral at *t=a* since *u(t–a)* is *0* for *t<a*, thus

$$L\{(u-a)\} = \frac{e^{-as}}{s} \qquad ,(s > 0) \qquad (10)$$

2.9- Table of Laplace Transforms

	f(t)	*L(f)=F(s)*

Some standard results of several functions *f(t)* and its Laplace Transform *L{f(t)}* which can be easily proven using Laplace Transform definition.

	f(t)	L(f)=F(s)
1	1	$\frac{1}{s}$
2	t	$\frac{1}{s^2}$
3	t^2	$\frac{2!}{s^3}$
4	t^n	$\frac{n!}{s^{n+1}}$
5	e^{at}	$\frac{1}{s-a}$
6	te^{at}	$\frac{1}{(1-s)^2}$
7	*Cos(wt)*	$\frac{s}{s^2+w^2}$
8	*Sin (wt)*	$\frac{w}{s^2+w^2}$
9	*Cosh (at)*	$\frac{s}{s^2-a^2}$
10	*Sinh (at)*	$\frac{a}{s^2+a^2}$
11	$e^{at}\cos(wt)$	$\frac{s-a}{(s-a)^2+w^2}$
12	$e^{at}\sin(wt)$	$\frac{w}{(s-a)^2+w^2}$

2.10− Systems of ODE's.

To solve a system of ODEs we could use Laplace Transform as we going to explain in the following example.

Example 6: if we have the following first order system

$$y_1' = 7y_1 - 3y_2 \quad , \quad y_2' = 10y_1 - 4y_2 \quad where\ y_1(0) = 7, y_2(0) = 12$$

taking Laplace Transform for the two sides of the system, writing $Y_1 = L\{y_1\}, Y_2 = L\{y_2\}$ and from the Laplace transform of derivatives we get

$$s\,Y_1 - 7 = 7\,Y_1 - 3\,Y_2$$

$$s\,Y_2 - 12 = 10\,Y_1 - 4\,Y_2$$

By gathering Y_1 and Y_2 terms we get

$$(s-7)Y_1 + 3\,Y_2 = 7$$
$$(s-10)Y_2 + 4\,Y_2 = 12$$

By solving the system algebraically

$$Y_1 = \frac{1}{s-1} + \frac{6}{s-2} \quad \rightarrow \quad Y_1 = L\{\,y_1\} = L\{e^t + 6e^{2t}\}$$

and so $\quad y_1 = e^t + 6e^{2t}$

Similarly

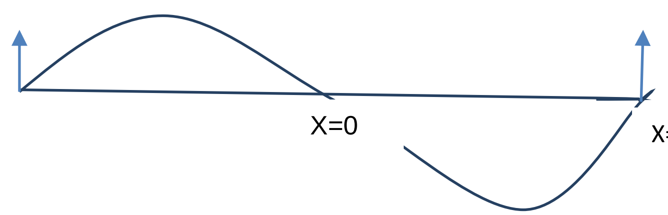
$$Y_2 = \frac{2}{s-1} + \frac{10}{s-2} \quad \rightarrow \quad Y_2 = L\{\, y_2\} = L\{2e^t + 10e^{2t}\}$$

and there for $\quad y_2 = 2e^t + 10e^{2t}$

So, the solution of our system is

$$y_1 = e^t + 6e^{2t} \quad \text{and} \quad y_2 = 2e^t + 10e^{2t}$$

2.11−An application of Laplace transform in physics is the harmonic vibration of a flexible rod. Considering a flexible rod with a cross−section parallel to the y z plane as in the figure below



Which has the equation $\quad EI\frac{d^4F}{dx^4} - MW^2F = 0 \quad (1)$

with the initial conditions

$$F(0) = 0, F(L) = 0\,; F''(0) = 0, F''(L) = 0 \qquad (2)$$

Where in the last equation E symbolizes the Bing modulus of elasticity, and I is the moment of inertia and M is the mass per unit length and W is the angular frequency.

Dividing equation (2) by EI, and let $\alpha^4 = \frac{MW^2}{EI}$we get

$$\frac{d^4F}{dx^4} - \alpha^4F = 0 \qquad (3)$$

By applying Laplace transform to equation (3) we have

$$s^4F(s) - s^3F(0) - s^2F'(0) - sF''(0) - F'''(0) - \alpha^4F(s) = 0 \qquad (4)$$

By using the initial conditions for $F(0)\,, F''(0)$ we get

$$F(s)\{s^4 - \alpha^4\} = s^2F'(0) + F'''(0) \qquad (5)$$

$$F(s) = \frac{s^2F'(0) + F'''(0)}{s^4 - \alpha^4} \qquad (6)$$

By taking the inverse of Laplace transform of (6)

$$L^{-1}\{F(s)\} = F'(0)L^{-1}\left\{\frac{s^2}{s^4-\alpha^4}\right\} + F'''(0)L^{-1}\left\{\frac{1}{s^4-\alpha^4}\right\} \qquad (7)$$

By using partial fraction, we have

$$F(x) = F'(0)\left(\frac{1}{2\alpha}\sinh\alpha\, x + \frac{1}{2\alpha}\sin\alpha x\right) + F'''(0)\left(\frac{1}{2\alpha^3}\sinh\alpha\, x + \frac{1}{2\alpha^3}\sin\alpha x\right)$$

$$\therefore\ F(x) = \left(\frac{F'(0)}{2\alpha} + \frac{F'''(0)}{2\alpha^3}\right)\sinh\alpha\, x + \left(\frac{F'(0)}{2\alpha} - \frac{F'''(0)}{2\alpha^3}\right)\sin\alpha x$$

$$let\left(\frac{F'(0)}{2\alpha} + \frac{F'''(0)}{2\alpha^3}\right) = A_1\,, \qquad \left(\frac{F'(0)}{2\alpha} - \frac{F'''(0)}{2\alpha^3}\right) = A_2$$

$$\therefore\ F(x) = A_1\sinh\alpha\, x + A_2\sin\alpha x \qquad (8)$$

To find $A_1 and\ A_2$ we use the following conditions $F(L) = 0\,; F''(L) = 0$

So, $\quad 0 = A_1\sinh\alpha\, l + A_2\sin\alpha l \qquad (9)$

$$0 = A_1\alpha^2\sinh\alpha\, l - A_2\alpha^2\sin\alpha l$$

$$0 = A_1\sinh\alpha\, l - A_2\sin\alpha l \qquad (10)$$

By adding equation (9)and (10)we obtain $\quad 2A_1\sinh\alpha\, l = 0$

$$where\ \sinh\alpha\, l \neq 0\ \rightarrow A_1 = 0 \qquad (11)$$

By subtracting equation (9)and (10)we obtain $\quad 2A_2\sin\alpha\, l = 0$

We get solution other than the trivial one

$$A_2 \neq 0\ \rightarrow \sinh\alpha\, l = 0 \rightarrow\ \alpha l = n\pi\ \rightarrow \alpha = \frac{n\pi}{l} \qquad (12)$$

Substituting equation (11) and (12) in (8) we have

$$F_n(x) = A_n sin\frac{n\pi x}{l} \qquad (13)$$

To find the angular frequency we use equation (12)

$$\therefore \alpha^4 = \frac{MW^2}{EI} = \frac{n^4\pi^4}{l^4}$$

$$\therefore W^2 = \frac{n^4\pi^4}{l^4}\frac{EI}{M}$$

$$W_n = \frac{n^2\pi^2}{l^2}\ \sqrt{\frac{EI}{M}}$$

3- Conclusion:

By Laplace Transform we can convert the function that we dealing with from its complex form to another form, which may be simpler and easier to deal with than the original function, which makes Laplace Transform a strong device of mathematics and other areas. In this paper overall we discussed what the Laplace Transform is. Also, the main use of Laplace Transform which is the change of the function from the time domain into its frequency domain equivalent was explained. The main Laplace Transform properties and one special function were also described. And showed how Laplace Transform could solve different types of problems.

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