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حساب المعاملات لفصل جديد من الدوال التحليلية ذات المعاملات السالبة

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The Coefficient Estimates for A New Class of Analytic Functions with Negative Coefficients

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الملخص:

في هذا البحث تم دراسة فصل جديد للدوال التحليلية على قرص الوحدة المفتوح. كذلك تم تعريفه بشكل أساسي بواسطة معامل المشتقة المعمم وتم الحصول على حساب المعاملات، و اشتقاق خواص أخرى يتميز بها هذا الفصل . بالإضافة إلى ذلك، تم دراسة حاصل ضرب هاردمارد لهذه الدوال.

الكلمات الدالة: الدوال الاحادية ، الدوال التحليلية ، المؤثر التفاضلي المعمم ، حاصل ضرب هادمارد ، متسلسلة القوى المعيارية.

Abstract

This paper introduces a new class in the open unit disc of analytic functions. It is mainly defined by the generalized derivative operator. A coefficient estimates is obtained, and other properties are derived. Additionally, Hadamard products (or convolution) of functions respective to the class are also included.

Keywords: Analytic function, generalized derivative operator, Hadamard products, normalized power series, univalent functions.

1 Introduction:

Let $\mathcal{A}$ denote the class of functions $f(z)$ given by the normalized power series

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \mathbb{U}),$$

Where a_k is a complex number and $f(z)$ is functions in the open unit disk $\mathbb{U} = \{z: |z| < 1; z \in \mathbb{C}\}$.

This is analytic in $\mathbb{U}$ satisfying the usual normalization conditions given by

$$\grave{f}(0) = 1 + f(0) = 1\,.$$

The Hadamard product (also known as convolution) for two analytic functions f as is in equation (1) and

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \quad , \qquad (z \in \mathbb{D}).$$

is provided by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \ ,$$

We also denote by T the subclass of $\mathcal{S}$ consisting of functions of the form:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \qquad\qquad (1)$$

This subclass was first established and investigated by Silverman [19] (also see [11], [12]).

For a function $f(z) \in \mathrm{T}$, the Jackson's q–derivative [16] ($0 < q < 1$), which is already introduced in several earlier investigations (see, for example [8,9,16].

$$\nabla_{\mathrm{q}} f(z) = 1 - \sum_{k=2}^{\infty} [\mathrm{k}]_{\mathrm{q}} a_k z^{k-1},$$

Where,

$$[\mathrm{k}]_{\mathrm{q}} = \frac{1 - q^k}{1 - q} \quad , \quad [0]_{\mathrm{q}} = 0\,.$$

As $q \to 1-$, $[k]_q$ = k and $\nabla_q f(z) = f'(z)$.

Motivated by the importance of studying the applications of quantum calculus in the physical and mathematical sciences, the authors in [6] introduced the generalized derivative operator given by

Definition 1 ([6]).

For $f \in \mathcal{A}$ the operator $I^m(\lambda_1, \lambda_2, l, n)$ is defined by $I^m(\lambda_1, \lambda_2, l, n) : \mathcal{A} \longrightarrow \mathcal{A}$

$I^m(\lambda_1, \lambda_2, l, n) f(z) = \phi^m(\lambda_1, \lambda_2, l)(z) * R^n f(z)$,$(z \in \mathbb{U})$

Where $m \in \mathbb{N}_0 = \{0,1,2, \ldots.\}$ and $\lambda_2 \geq \lambda_1 \geq 0$, $l \geq 0$, and $R^n f(z)$ denotes the Ruscheweyh derivative operator [14] ,and given by

$$R^n f(z) = z + \sum_{k=2}^{\infty} c(n,k) a_k z^k \, , (n \in \mathbb{N}_0, z \in \mathbb{U}),$$

Where

$$c(n,k)=\frac{(n+1)_{k-1}}{(1)_{k-1}}$$

If f is given by (1), then we easily find that

$$I^m(\lambda_1,\lambda_2,l,n)f(z)=z+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}\big(1+\lambda_2(k-1)\big)^m}c(n,k)a_kz^k,$$

where $n,m\in\mathbb{N}_0=\{0,1,2,\dots.\}$, $\lambda_2\geq\lambda_1\geq 0$, $l\geq 0$.

Special cases of this operator includes:

- the Ruscheweyh derivative operator[14] in the cases:

$I^1(\lambda_1,0,l,n)\equiv I^1(\lambda_1,0,0,n)\equiv I^1(0,0,l,n)\equiv I^0(0,\lambda_2,0,n)$
$\equiv I^0(0,0,0,n)\equiv I^{m+1}(0,0,l,n)\equiv I^{m+1}(0,0,0,n)\equiv R^n$,

- the Salagean derivative operator [15]

$$I^{m+1}(1,0,0,0)\equiv S^n,$$

- The generalized Ruscheweyh derivative operator [17]:

$$I^2(\lambda_1,0,0,n)\equiv R^n_\lambda\ ,$$

- The generalized Salagean derivative operator introduced by Al−Oboudi [2]:

$$I^{m+1}(\lambda_1,0,0,0)\equiv S^n_\beta\ ,$$

- The generalized Al−Shaqsi and Darus derivative operator [3]:

$$I^{m+1}(\lambda_1,0,0,n)\equiv R^n_{\lambda,\beta}\ ,$$

- The Al−Abbadi and Darus generalized derivative operator [4]:

$$I^m(\lambda_1,\lambda_2,0,n)\equiv \mu^{n,m}_{\lambda_1,\lambda_2},$$

Finally,

- The Catas derivative operator [10]:

$$I^m(\lambda_1,0,l,n)\equiv I^m(\lambda,\beta,l).$$

Using simple computation one obtains the next result .

$$(1+l)I^{m+1}(\lambda_1,\lambda_2,l,n)f(z)=(1+l-\lambda_1)[I^m(\lambda_1,\lambda_2,l,n)*\phi^1(\lambda_1,\lambda_2,l)(z)]f(z)$$
$$+\lambda_1 z[(I^m(\lambda_1,\lambda_2,l,n)*\phi^1(\lambda_1,\lambda_2,l)(z)]'.$$

Where $(z\in\mathbb{U})$ and $\phi^1(\lambda_1,\lambda_2,l)(z)$ an analytic function and form (2) given by

$$\phi^1(\lambda_1,\lambda_2,l)(z)=z+\sum_{k=2}^{\infty}\frac{1}{(1+\lambda_2(k-1))}z^k.$$

Definition 2:

Let $\lambda_2\geq\lambda_1\geq 0$, $l\geq 0$, $0\ \leq\ \gamma\ <\ 1$, and $f\in\mathrm{T}$, such that $I^m(\lambda_1,\lambda_2,l,n)f(z)$ for $z\in\mathbb{U}$. We say that $f\in\emptyset^m_q(\lambda_1,\lambda_2,l,n,\gamma)$ if and only if

$$\emptyset_q^m(\lambda_1,\lambda_2,l,n,\gamma) \ =\left\{f\in\mathcal{A}:Re\left\{\frac{z\nabla_q(I^m(\lambda_1,\lambda_2,l,n)f(z))}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\right\}>\gamma\ ,\right\}$$

Now, we define the class given by $\emptyset_q^m(\lambda_1,\lambda_2,l,n,\gamma)$.

The aim of this paper is to examine various properties with respect to functions f that belong to this class.

2 Coefficient Estimates

Theorem 1:

The function $f\in\emptyset_q^m(\lambda_1,\lambda_2,l,n,\gamma)$ if and only if

$$\sum_{k=2}^{\infty}([k]_q-\gamma)\Psi_{q,k}^m(\lambda_1,\lambda_2)a_k\leq 1-\gamma. \qquad (2)$$

Proof:

Assume that (2) holds true. It is sufficient to show that

$$\left|\frac{z\nabla_q I^m(\lambda_1,\lambda_2,q)f(z)}{I^m(\lambda_1,\lambda_2,q)f(z)}-1\right|=\left|\frac{\sum_{k=2}^{\infty}(1-[k]_q)\Psi_{q,k}^m(\lambda_1,\lambda_2)a_k z^k}{z-\sum_{k=2}^{\infty}\Psi_{q,k}^m(\lambda_1,\lambda_2)\,a_k}\right|.$$

This last expression is bounded above by $1-\gamma$. then $f\in\emptyset_q^m(\lambda_1,\lambda_2,\gamma)$

Now, let $f\in\emptyset_q^m(\lambda_1,\lambda_2,\propto)$, then

$$\mathrm{Re}\left\{\frac{z\nabla_q I^m(\lambda_1,\lambda_2,q)f(z)}{I^m(\lambda_1,\lambda_2,q)f(z)}\right\}=\mathrm{Re}\left\{\frac{z-\sum_{k=2}^{\infty}[k]_q\,\Psi_{q,k}^m(\lambda_1,\lambda_2)a_k z^k}{z-\sum_{k=2}^{\infty}\Psi_{q,k}^m(\lambda_1,\lambda_2)\,a_k}\right\}>\propto.$$

Choose values of z on real axis so that $\frac{z\nabla_q I^m(\lambda_1,\lambda_2,q)f(z)}{I^m(\lambda_1,\lambda_2,q)f(z)}$ is real.

Letting $z\to 1^-$through real values, we have

$$1-\sum_{k=2}^{\infty}[k]_q\,\Psi_{q,k}^m(\lambda_1,\lambda_2)a_k z^k\geq\gamma-\sum_{k=2}^{\infty}\propto\Psi_{q,k}^m(\lambda_1,\lambda_2)\,a_k z^k.$$

Thus we obtain

$$\sum_{k=2}^{\infty}([k]_q-\gamma)\Psi_{q,k}^m(\lambda_1,\lambda_2)a_k\leq 1-\gamma.$$

Which is (1). Hence the theorem is holds.

Corollary 1:

If $f\in\emptyset_q^m(\lambda_1,\lambda_2,l,n,\gamma)$, such that

$$f(z)=z-\frac{1-\gamma}{([k]_q-\gamma)\Psi_{q,k}^m(\lambda_1,\lambda_2,l,n)}z^k \qquad (2)$$

Then we have

$$a_k\leq\frac{1-\gamma}{([k]_q-\gamma)\Psi_{q,k}^m(\lambda_1,\lambda_2,l,n)} \qquad (3)$$

Definition 3:

Let $\emptyset_q^m(\lambda_1,\lambda_2,l,n,\gamma,d_n)$ be the subclass of $\emptyset_q^m(\lambda_1,\lambda_2,l,n,\gamma)$ consisting of functions of the form

$$f(z)=z-\sum_{i=2}^{n}\frac{d_i(1-\gamma)}{([i]_q-\gamma)\Psi_{q,k}^m(\lambda_1,\lambda_2,l,n)}z^i-\sum_{k=n+1}^{\infty}a_k z_k. \qquad (4)$$

Where $0\le d_i\le 1$ and $\sum_{i=2}^{n}d_i\le 1$.

Theorem 2

Let $f(z)\in\emptyset_q^m(\lambda_1,\lambda_2,l,n,\gamma)$, Then $f(z)\in\emptyset_q^m(\lambda_1,\lambda_2,l,n,\gamma,d_n)$ if and only if

$$\sum_{k=n+1}^{\infty}([k]_q-\gamma)\Psi_{q,k}^m(\lambda_1,\lambda_2,l,n)a_k\le(1-\gamma)(1-\sum_{i=2}^{n}d_i). \qquad (5)$$

Proof:

Assume that

$$a_i=\frac{d_i(1-\gamma)}{([i]_q-\gamma)\Psi_{q,k}^m(\lambda_1,\lambda_2,l,n)}, \quad \text{for } i=2,3,\dots,n$$

By substituting the value of a_i , we obtain

$$\sum_{i=2}^{n}d_i+\sum_{k=n+1}^{\infty}\frac{([k]_q-\gamma)\Psi_{q,k}^m(\lambda_1,\lambda_2,l,n)}{(1-\gamma)}a_k\le 1.$$

Then the equality (5) is holds.

Now, if we assume that (5) is true, then

$$f(z)=z-\sum_{i=2}^{n}\frac{d_i(1-\gamma)}{([i]_q-\gamma)\Psi_{q,k}^m(\lambda_1,\lambda_2,l,n)}z^i-\sum_{k=n+1}^{\infty}\frac{(1-\gamma)\sum_{i=2}^{n}d_i}{([k]_q-\gamma)\Psi_{q,k}^m(\lambda_1,\lambda_2,l,n)}z^i. \qquad (6)$$

Corollary 2

If $f(z)\in\emptyset_q^m(\lambda_1,\lambda_2,l,n,\gamma,d_n)$, and satisfied equations (4) and (6), then

$$a_{\mathrm{k}}\le\frac{(1-\gamma)(1-\sum_{i=2}^{n}d_i)}{\big([k]_q-\gamma\big)\Psi_{q,k}^m(\lambda_1,\lambda_2,l,n)}, \qquad k\ge n+1 \quad .$$

Theorem 3:

If $f(z)\in\emptyset_q^m(\lambda_1,\lambda_2,l,n,\gamma,d_n)$, then

$$\sum_{\mathrm{k=n+1}}^{\infty}[k]_q a_{\mathrm{k}}\le\frac{[n+1]_q(1-\gamma)(1-\sum_{i=2}^{n}d_i)}{([n+1]_q-\gamma)\Psi_{q,k}^m(\lambda_1,\lambda_2,l,n+1)}.$$

Proof :

Let $f(z)\in\emptyset_q^m(\lambda_1,\lambda_2,l,n,\gamma,d_n)$, then , from (5), we have

$$\big([n+1]_q-\gamma\big)\Psi_{q,k}^m(\lambda_1,\lambda_2,l,n+1)\sum_{\mathrm{k=n+1}}^{\infty}a_{\mathrm{k}}\le(1-\gamma)\left(1-\sum_{i=2}^{n}d_i\right),$$

Then

$$\sum_{\mathrm{k=n+1}}^{\infty} a_{\mathrm{k}} \leq \frac{(1-\gamma)(1-\sum_{i=2}^{n} d_i)}{\left([n+1]_q-\gamma\right)\Psi_{q,k}^{m}(\lambda_1,\lambda_2,l,n+1)}$$

So,

$$\Psi_{q,k}^{m}(\lambda_1,\lambda_2,l,n+1)\sum_{\mathrm{k=n+1}}^{\infty}[k]_q a_{\mathrm{k}} \leq (1-\gamma)\left(1-\sum_{i=2}^{n} d_i\right)+\gamma\Psi_{q,k}^{m}(\lambda_1,\lambda_2,l,n+1)\sum_{\mathrm{k=n+1}}^{\infty} a_{\mathrm{k}}\,.$$

Which can written in the form:

$$\Psi_{q,k}^{m}(\lambda_1,\lambda_2,l,n+1)\sum_{\mathrm{k=n+1}}^{\infty}[k]_q a_{\mathrm{k}} \leq (1-\gamma)\left(1-\sum_{i=2}^{n} d_i\right)+\gamma\frac{(1-\gamma)(1-\sum_{i=2}^{n} d_i)}{([n+1]_q-\gamma)}.$$

Then

$$\sum_{\mathrm{k=n+1}}^{\infty}[k]_q a_{\mathrm{k}} \leq \frac{[n+1]_q(1-\gamma)(1-\sum_{i=2}^{n} d_i)}{\left([n+1]_q-\gamma\right)\Psi_{q,k}^{m}(\lambda_1,\lambda_2,l,n+1)}.$$

Theorem 4:

Let the function $f(z) \in \emptyset_q^m(\lambda_1,\lambda_2,l,n,\gamma,d_n)$, such that

$$f(z)=z-z^2\sum_{i=2}^{n}\frac{d_i(1-\gamma)}{\left([i]_q-\gamma\right)\Psi_{q,i}^{m}(\gamma_1,\gamma_2)}-\sum_{k=n+1}^{\infty}\frac{(1-\gamma)(1-\sum_{i=2}^{n} d_i)}{\left([k]_q-\gamma\right)\Psi_{q,k}^{m}(\gamma_1,\gamma_2)}z^{n+1},$$

Then

$$|z|-|z|^2\sum_{i=2}^{n}\frac{d_i(1-\gamma)}{\left([i]_q-\gamma\right)\Psi_{q,k}^{m}(\lambda_1,\lambda_2,l,n)}-\sum_{k=n+1}^{\infty}\frac{(1-\gamma)\sum_{i=2}^{n} d_i}{\left([k]_q-\gamma\right)\Psi_{q,k}^{m}(\lambda_1,\lambda_2,l,n)}|z|^{n+1}$$
$$\leq |f(z)| \leq$$
$$|z|+|z|^2\sum_{i=2}^{n}\frac{d_i(1-\gamma)}{([i]_q-\gamma)\Psi_{q,k}^{m}(\lambda_1,\lambda_2,l,n)}+\sum_{k=n+1}^{\infty}\frac{(1-\gamma)\sum_{i=2}^{n} d_i\,|z|^{n+1}}{([k]_q-\gamma)\Psi_{q,k}^{m}(\lambda_1,\lambda_2,l,n)}, \quad (15)$$

Proof:

By applying the triangle inequality and some other properties of inequalities for the equation ,

we can easily deduce the proof as follows:

$$|f(z)|=\left|z-\sum_{i=2}^{n}\frac{d_i(1-\gamma)}{([i]_q-\gamma)\Psi_{q,k}^{m}(\lambda_1,\lambda_2,l,n)}z^i-\sum_{k=n+1}^{\infty}a^k z^{n+1}\right|$$
$$\leq |z|+|z|^2\sum_{i=2}^{n}\frac{d_i(1-\gamma)}{([i]_q-\gamma)\Psi_{q,k}^{m}(\lambda_1,\lambda_2,l,n)}+|z|^{n+1}\sum_{k=n+1}^{\infty}a^k,$$

and

$$|f(z)|=\left|z-\sum_{i=2}^{n}\frac{d_i(1-\gamma)}{([i]_q-\gamma)\Psi_{q,k}^{m}(\lambda_1,\lambda_2,l,n)}z^i-\sum_{k=n+1}^{\infty}a^k z^{n+1}\right|$$

$$\geq |z| - |z|^2 \sum_{i=2}^{n} \frac{d_i(1-\gamma)}{([i]_q - \propto)\Psi_{q,k}^m(\lambda_1,\lambda_2,l,n)} - |z|^{n+1} \sum_{k=n+1}^{\infty} a^k.$$

Many other work on analytic functions related to derivative operator and integral operator can be read in [1,5,7,16, 18] .

Conclusion:

In this paper, we used new results are related to the class $\emptyset_q^m(\lambda_1,\lambda_2,l,n,\gamma)$ of analytic function in $\mathbb{U}$ and obtained a new class of analytic functions which defined on the open unit disc by using a generalized derivative operator .Also, we may obtain the Hadamard convolutions of functions .In our future paper , with the aid of q−calculus, we will investigate a same new subclass of analytic functions involving the modified q derivative operator. The concept outlined in this article can be employed to easily study a large range of analytic and univalent functions linked to several theorem. This may open numerous new lines of inquiry into the Geometric Function,theory of Complex Analysis and appropriate areas.

References

[1] Abufares ,F .and Amer .A , (2023) , Some Applications of Fractional Differential Operators in the Field of Geometric Function .Theory,Conference on basic sciences and their applications

[2] Al−Oboudi, F.M. (**2004**) , On univalent functions defined by a generalized Sălăgean operator. International Journal of Mathematics and Mathematical Sciences,. p. 1429−1436.

[3] Al−Shaqsi, K. and Darus . M , (2008), Differential subordination with generalized derivative operator. Int. J. Comp. Math. Sci,**2** p. 75−78.

[4] Al−Abbadi, M . and Darus . M, (2009) , Differential subordination for new generalised derivative operator. Acta Universitatis Apulensis. Mathematics−Informatics, **20**. p. 265−280.

[5] Alabbar. N, Darus. M & Amer.A. (2023), Coefficient Inequality and Coefficient Bounds for a New Subclass of Bazilevic Functions. Journal of Humanitarian and Applied Sciences, **8** −496−506.

[6]Amer, A.A. and Darus. M , (2011), On some properties for new generalized derivative operator. Jordan Journal of Mathematics and Statistics (JJMS), **4** −(2): p. 91−101.

[7] Amer.A , Darus , M and Alabbar .N .(2024) , Properties For Generalized Starlike and Convex Functions of Order , Fezzan University scientific Journal.

[8] Annby. M. H and Mansour. Z. S, (2012) , q−Fractional Calculus Equations. Lecture Notes in Mathematics., Vol. **2056,** Springer, Berlin.

[9] Aouf. M. K. , Darwish. H. E and S˘al˘agean . G. S, (2001), , On a generalization of starlike functions with negative coefficients, Math., Tome **66** − no. 1, 3–10.

[10] Catas, A. and Borsa .E , (2009). On a certain differential sandwich theorem associated with a new generalized derivative operator. General Mathematics, **17**−p. 83−95

[11] Frasin . B.A (2006), Family of analytic functions of complex order, Acta Math. Acad. Paedagogicae Ny , **22** −no. 2, 179–191.

[12] Ismail. M.E, Merkes. M.E , and David. M.E, (1990), A generalization of starlike functions, Transit. Complex Variables, Theory Appl. **61** ,77–84.

[13] Jackson. F. H, (1908), On q–functions and a certain difference operator, Trans. R. Soc. Edinb., **46**−253–281.

[14] Ruscheweyh, S. (1975). New criteria for univalent functions. Proceedings of the American Mathematical Society, **49** −(1): p. 109−115.

[15] Salagean, G.S. (1981), Subclasses of univalent functions, Complex analysis−fifth Romanian−Finnish seminar, **1** , 362−372. Lecture Notes in Math, 1983.

[16] SALEH .Z. M. , MOSTAFA. A. O. (2022), Class of analytic univalent functions with fixed finite negative coefficients defined by q−analogue difference operator, Jordan Journal of Mathematics and Statistics (JJMS), 15(4A), 2022, pp 955− 965

[17] Shaqsi, K. and Darus . M , (2008) ,An operator defined by convolution involving the polylogarithms functions. Journal of Mathematics and Statistics, **4**. (1): p. 46.

[18] Shmella. E.K and Amer. A.A , (2023) , Estimation of the Bounds of Univalent Functional of Coefficients Apply the Subordination Method, The Academic Open Journal Of Applied And Human Sciences ,**5**,(2709−3344), issue (1) .

[19] Silverman. H , (1975), Univalent functions with negative coefficients, Proc. Amer. Math. Soc. **51** ,no. 1, 109–116.