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Analytic study of Fourier Series Convergence with applications to the One Dimensional Heat Equation

Iman Omran Nassr*
Department of mathematics, faculty of Education, Gulf of Sidra University,Bin Jawad, libya
imannassr@gsu.edu.ly

دراسة تحليلية لتقارب متسلسلة فورييه مع تطبيقات على معادلة الحرارة أحادية البعد

إيمان عمران نصر*
قسم الرياضيات، كلية التربية، جامعة خليج السدرة، بن جواد ، ليبيا

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الملخص:
تستكشف هذه الورقة البحثية الخصائص الرياضية والتطبيقات العملية لمتسلسلات فورييه، مع التركيز على الانتقال من تقريب الدوال الدورية إلى حل المعادلات التفاضلية الجزئية. من خلال تحليل دالة الموجة المربعة، نُبين كفاءة استخدام توسعات نصف المدى والامتدادات الفردية في تبسيط حساب معاملات فورييه. علاوة على ذلك، تُعمم الدراسة هذه النتائج الرياضية لحل معادلة الحرارة أحادية البعد باستخدام طريقة فصل المتغيرات في النهاية، تُظهر هذه الدراسة فعالية طرق متسلسلات فورييه في حل مسائل القيم الحدية ونمذجة انتشار الحرارة في المجالات المحدودة.

الكلمات الدالة: متسلسلات فورييه، المعادلات التفاضلية الجزئية، معادلة الحرارة، الموجة المربعة، شروط ديريشليه الحدية.

Abstract
This paper explores the mathematical properties and practical application of the Fourier Series, focusing on the transition from periodic function approximation to solving Partial Differential Equations (PDEs). Through the analysis of a square wave function, we demonstrate the efficiency of using half-range expansions and odd extensions in simplifying the calculation of Fourier coefficients. Furthermore, the study extends these mathematical results to solve the one-dimensional heat equation using the method of separation of variables. Ultimately, this study demonstrates the effectiveness of Fourier series methods in solving boundary value problems and modeling heat diffusion in finite domains.

Keywords: Fourier Series, Partial Differential Equations (PDEs), Heat equation, Square wave, Dirichlet boundary conditions.

1- Introduction

The evolution of mathematical physics was profoundly impacted by Joseph Fourier's proposal that a broad class of periodic functions can be represented as a sum of trigonometric functions. This concept, known as the Fourier series, has become an indispensable tool in engineering, signal processing, and physics.

The prime challenge in many physical systems is representing discontinuous or non-smooth initial states. By utilizing the symmetry properties of functions such as even and odd extension, we can tailor Fourier representations to satisfy specific boundary conditions. This paper begins by deriving the Fourier coefficients for a square wave on a specific interval, highlighting the computational advantage of half-range sine expansions. Subsequently, we demonstrate the application of these coefficients in solving the Heat equation, illustrating the transition from a static mathematical series to a dynamic physical solution that describes energy distribution over time.

2- General Form of PDEs

If u is an unknown function depending on variables x and y, a second-order PDE can be expressed as

$$F\left(x, y, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y}\right) = 0$$

Partial differential equations of the form $Au_{xx} + Bu_{xy} + Cu_{yy} + \cdots = 0$ are generally classified into three main types based on the discriminant $B^2 - AC$:

1- Elliptic Equation($B^2 - AC < 0$)s: such as Laplace's equation.
2- Parabolic Equations($B^2 - AC = 0$): such as the Heat Equation.
3- Hyperbolic Equations($B^2 - AC > 0$): such as the Wave Equation.

The classification of PDE's depends on the physical interpretation of the problem.

3- Fourier Series

Fourier series is considered one of the most important mathematical techniques used in analyzing periodic function. The French mathematician Joseph Fourier introduced this method during his study of heat conduction in solid bodies. The fundamental idea of Fourier Series is that any periodic and integrable function can be represented as an infinite sum of sine and cosine functions. This representation simplifies complicated mathematical problems and allows them to be solved more efficiently.

A periodic function is a function defined for all real x, satisfying the fundamental property:

$$f(x+p) = f(x), \forall x$$

where p is the fundamental positive period.

Consequently, it holds that

$$f(x+np) = f(x), \qquad \forall x \text{ and for any } n \in \mathbb{Z}$$

Simple functions of period 2π are

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos nx, \sin nx$$

The fundamental periodic behavior and geometric profiles of the basic components of this trigonometric system, namely $\sin(x)$ and $\cos(x)$, on the interval $[-\pi, \pi]$ are illustrated below in Figure 1.

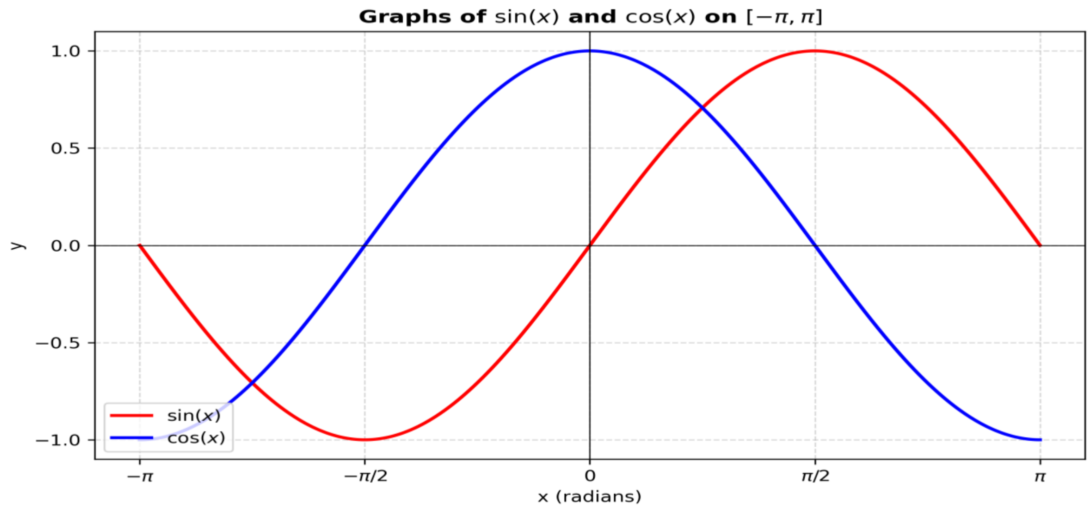

Figure 1

And in the interval $-\pi \leq x \leq \pi$ we have

$$\int_{-\pi}^{\pi} cos\, nx\, cos\, mx = 0 \qquad , n \neq m$$

$$\int_{-\pi}^{\pi} sin\, nx\, sin\, mx = 0 \qquad , n \neq m$$

$$\int_{-\pi}^{\pi} sin\, nx\, cos\, mx = 0 \qquad , \text{for all integers } n, m$$

The Fourier Series of a function $f\ (x)$ defined on the interval$(-\pi, \pi)$ is written as:

(1) $$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Where a_n are the cosine coefficients, b_n are the sine coefficients and π represents half the period of the function.
The coefficients of (1) are so-called Fourier coefficients of $f(x)$, and they are calculated using the following:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

(2) $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx, \quad n = 1,2,3, \dots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx, \quad n = 1,2,3, \dots$$

Example 1: Find the Fourier series of the given function of a periodic rectangular wave:

$$f(x) = \begin{cases} -1, & if \quad -\pi < x < 0 \\ 1, & if \quad 0 < x < \pi \end{cases}, \qquad f(x + 2\pi) = f(x)$$

Solution

Using the Fourier Series formula

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

we have to find the Fourier coefficients by using the Fourier coefficient formulas in (2) we get

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-1)dx - \int_{0}^{\pi} dx\right] = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\cos nx\, dx - \int_{0}^{\pi} \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-\sin nx}{n}\bigg|_{-\pi}^{0} + \frac{\sin nx}{n}\bigg|_{0}^{\pi}\right]$$

$$= \frac{1}{n\pi}[-\sin(0) + \sin(-n\pi) + \sin(n\pi) - \sin(0)] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\sin nx\, dx + \int_{0}^{\pi} \sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos nx}{n}\bigg|_{-\pi}^{0} - \frac{\cos nx}{n}\bigg|_{0}^{\pi}\right]$$

$$= \frac{1}{n\pi}[\cos(0) - \cos(-n\pi) - \cos(n\pi) + \cos(0)]$$

$$= \frac{1}{n\pi}[1 - \cos(-n\pi) - \cos(n\pi) + 1] = \frac{2}{n\pi}[1 - \cos(n\pi)]$$

$$= \begin{cases} \dfrac{4}{n\pi} & if\ n\ is\ odd \\ 0 & if\ n\ is\ even \end{cases}$$

Therefore the Fourier Series is

$$a_0 + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \cdots + b_1\sin x + b_2\sin 2x + b_3\sin 3x + \cdots$$

$$= 0 + 0 + 0 + \cdots + \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x + \ldots$$

$$= \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi}\sin(2k-1)x$$

The first three partial sums are

$$s_1 = \frac{4}{\pi}\sin x\ , s_3 = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x\, , s_5$$

$$= \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x$$

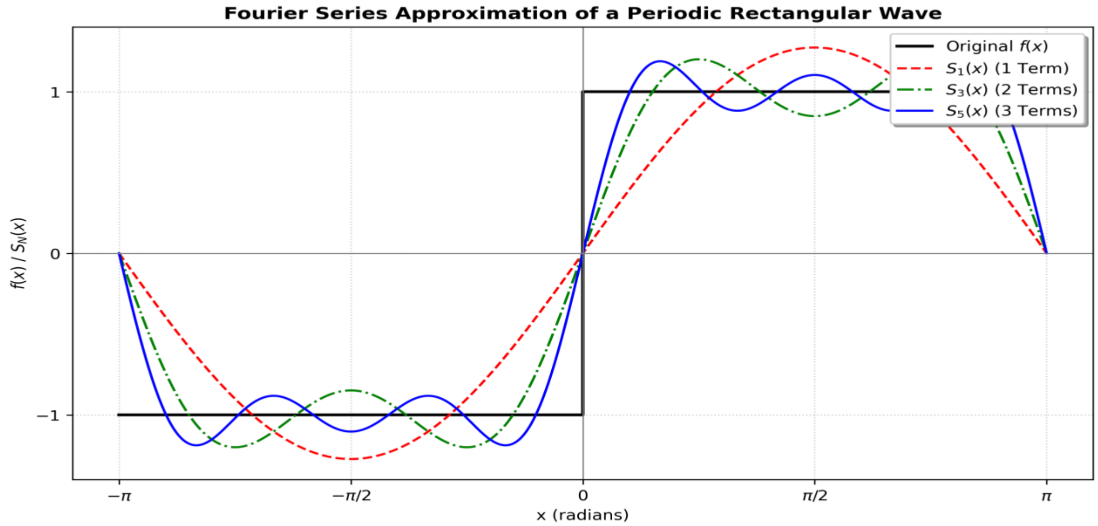

Figure 2

The numerical approximations illustrated in Figure 2 indicate that the Fourier Series converges to the function $f(x)$. At the discontinuity points $x = 0$, and $x = \pi$, the series converges to the average of the left-hand and the right-hand limits, which equals zero. This behavior is consistent with the convergence properties of the Fourier Series.

As observed in Figure 2, near the points of discontinuity$(x = 0, \pm\pi)$, the Fourier partial sums exhibt an overshoot and distinct oscillations known as the Gibbs Phenomenon. Even as the number of terms (n) approaches infinity, this overshoot does not disappear; instead, its width narrows while its amplitude remains constant at approximately 8.95% of the total jump magnitude. This highlights the inherent limitation of using continuous trigonometric series to approximate discontinuous functions.

4- Convergence and representation of Fourier Series

Fourier series offer a robust framework for representing periodic functions. In practical application, convergence is guaranteed by the Dirichlet conditions, ensuring that the series accurately reflects the physical or mathematical function under study.

To establish how a Fourier series behaves relative to its original function, we utilize the Dirichlet Convergence Theorem. This theorem guarantees that at any point where $f(x)$ is continuous, the series converges exactly to $f(x)$. Crucially, at any point of jump discontinuity x_0 , the series converges to the arithmetic mean of its left and right limits.The sufficient conditions for this convergence are defined as follows:

Theorem: Let $f(x)$ be a periodic function with period 2π. Assume that within the interval $[-\pi, \pi]$, the function satisfies the following conditions:

1. The function is piecewise continuous
2. The function has finite left-hand and right-hand derivatives at every point in the interval

Under these assumptions, the Fourier Series of $f(x)$converges according to the following results:

- At points where the function is continuous, the Fourier Series converges to the exact value of the function

$$s(x) = f(x)$$

- At points of discontinuity x_0, the Fourier Series converges to the arithmetic mean of the left-hand and the right-hand limits

$$s(x_0) = \frac{f(x_0^-) + f(x_0^+)}{2}$$

Where $f(x_0^-)$ denotes the left-hand limit, and (x_0^+) denotes the right-hand limit.
These convergence properties play an essential role in the applications of the Fourier Series to partial differential equations.

5- Generalizing Fourier Analysis
Arbitrary periods, odd and even functions, and half-Range techniques
5-1- Generalization to Arbitrary periods ($\boldsymbol{p = 2L}$)
Fourier series are commonly introduced for functions with period 2π. However, many physical and engineering problems involve intervals of length $2L$, such as Heat conduction and wave motion. To apply Fourier analysis to these problems, the series is generalized by scaling the variable x. Thus, a function defined on $[-L, L]$ can be represented by

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x \right) \tag{3}$$

with the Fourier coefficients:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx, \quad n = 1,2,3,\ldots \tag{4}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx, \quad n = 1,2,3,\ldots$$

5-2- Symmetry properties: Even and Odd functions

- Even function $f(-x) = f(x)$

The sine coefficient vanishes ($b_n = 0$), producing a Fourier Cosine Series.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x\,,$$

$$where \qquad a_0 = \frac{1}{L}\int_{0}^{L} f(x)dx, \qquad a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi}{L}x\,dx$$

- Odd function $f(-x) = -f(x)$

The cosine coefficient vanishes ($a_n = 0$), producing a Fourier Sine Series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x$$

$$where \qquad b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi}{L}x\,dx$$

5-3- Half-range expansion
When the function $f(x)$ is only defined on a specific interval $[0, L]$(the Half-range). To represent such a function using a Fourier Series, we employ a conceptual extension:

- Even extension: the function is extended symmetrically about the y-axis over the interval $[-L, L]$. This results in a Fourier cosine series, where all sine coefficients vanish.
- Odd extension: the function is extended antisymmetrically about the origin. This results in a Fourier sine series, where all cosine coefficients vanish.

This technique is extensively utilized in solving partial differential equations to satisfy specific boundary conditions; fixed chords use a sine series, while isolated boundary terms often use a cosine series.

Example 2: Find the Fourier series of the function:

$$f(x) = \begin{cases} -k, & if \ -1 < x < 0 \\ k, & if \ 0 < x < 1 \end{cases} \quad ,p = 2L \rightarrow L = 1$$

Solution

By applying the generalization of Fourier analysis to a function defined over a non-standard physical domain $[-1, 1]$. Here, the period is $p = 2L$ which implies $L = 1$.

We first examine the symmetry properties of the function $f(x)$.

Because $f(-x) = -f(x)$ the function is odd. Therefore, the constant coefficient and the cosine coefficient vanish.

$$a_0 = 0, a_n = 0$$

As a result, the expansion reduces to a Fourier Sine Series only.

Using the generalized formula for $L = 1$:

$$b_n = 2\int_0^1 k \sin n\pi x \, dx = 2k \left[\frac{-\cos n\pi x}{n\pi}\right]_0^1 = \frac{2k}{n\pi}(1 - \cos n\pi)$$

$$= \begin{cases} 0 \rightarrow for\ even\ n \\ \dfrac{4k}{n\pi} \rightarrow for\ odd\ n \end{cases}$$

$$f(x) = \frac{4k}{\pi} \sum_{n\ is\ odd}^{\infty} \frac{1}{n} \sin n\pi x$$

Note that in the previous solution, the transition from the full-period integral $[-1, 1]$ to the double half-period integral over $[0, 1]$ effectively employs the half-range sine expansion. This confirms that for an odd function, the standard Fourier process and the half-range approach optimize computational efficiency.

6- Application in Partial Differential Equations

The practical significance of the Fourier Series extends beyond functional approximation to the field of Partial Differential Equations (PDEs). By utilizing the principle of superposition and the method of separation of variables, complex physical phenomena can be decomposed into simpler harmonic components.

6-1 Thermal Diffusion in a Finite Rod

Let the temperature distribution $u(x, t)$ within a thin homogeneous metal rod of length $L = 1$ be considered. The thermal behavior of this system is described by the one-dimensional heat equation:

(5) $$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \quad ,0 < x < 1 \ ,t > 0$$

where α represents the thermal diffusivity of the rod's material.

with the boundary conditions

(6) $$u(0,t) = 0 \quad , \quad u(1,t) = 0$$

and the initial condition at $t = 0$ is given by the square wave function $f(x)$ defined in Example 2:

(7) $$u(x,0) = f(x) = \begin{cases} -k, & if \ -1 < x < 0 \\ k, & if \ 0 < x < 1 \end{cases}$$

Note that for the actual physical rod where $x \in [0,1]$, the initial temperature distribution is simply $u(x,0) = k$. However, to satisfy the homogeneous Dirichlet boundary conditions at the boundaries ($x = 0$ and $x = 1$), we mathematically construct an odd extension of the initial profile onto the interval $[-1,1]$. This extending procedure yields a periodic square wave configuration that is identical to the mathematical function defined and solved in Example 2, thereby allowing the direct application of its Fourier sine coefficients.
Applying the method of separation of variables by assuming a solution of the form

$$u(x,t) = X(x)T(t)$$

Substituting into the heat equation gives:

$$X(x)T'(t) = \alpha X''(x)T(t)$$

$$\Rightarrow \frac{T'(t)}{\alpha T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

where λ is a separation constant

This yields two ordinary differential equations (ODEs):

$$X''(x) + \lambda X(x) = 0$$

$$T'(t) + \alpha\lambda T(t) = 0$$

Applying the boundary conditions $X(0) = 0$ and $X(1) = 0$
The non-trivial solutions occur for

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

Since $L = 1$

$$\lambda_n = (n\pi)^2 \qquad for\ n = 1,2,3, \ldots$$

The corresponding eigenfunctions are:

$$X_n(x) = \sin n\pi x$$

Substituting $\lambda_n$into the time equation

(8) $$T_n(t) = C_n e^{-\alpha(n\pi)^2 t}$$

By the principle of superposition, the complete solution is expressed as the Fourier Sine Series

(9) $$u(x,t) = \sum_{n=1}^{\infty} b_n \sin n\pi x \, e^{-\alpha(n\pi)^2 t}$$

where we get the coefficients $b_n$by setting $t = 0$ and matching the initial temperature profile:

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin n\pi x = f(x)$$

This is exactly the Fourier Sine Series for the square wave on $[0,1]$, using the result from Example 2:

$$b_n = 2\int_0^1 k \sin n\pi x \, dx = \frac{2k}{n\pi}[1 - \cos n\pi]$$

which simplifies to:

$$b_n = \begin{cases} \frac{4k}{n\pi} & for\ odd\ n \\ 0 & for\ even\ n \end{cases}$$

So, the temperature evolution of the rod is described by the following equation:

(10) $$u(x,t) = \sum_{n\ is\ odd}^{\infty} \frac{4k}{n\pi} \sin n\pi x \ e^{-\alpha(n\pi)^2 t}$$

This example demonstrates how Fourier Series can be used to transform partial differential equations into a set of simple ODEs, providing an analytic solution to the one-dimensional heat conduction problem.

7- Conclusion

This paper has demonstrated that Fourier Series are more than just a tool for functional approximation; they are a fundamental bridge between abstract mathematics and physical reality. By analyzing the symmetry of functions, specifically through odd extension and half-range expansion, we can significantly simplify the computation of coefficients. These mathematical foundations directly enable the solution of complex Partial Differential Equations, such as the Heat Equation. The ability to decompose allows us to predict the dynamic behavior of systems, proving that Fourier analysis remains an indispensable pillar in engineering and physics.

References

1. Boyce, W. E., DiPrima, R. C., & Meade, D. B. (2017). Elementary Differential Equations and Boundary Value Problems (11th ed.). John Wiley & Sons.

2. Churchill, R. V., & Brown, J. W. (1978). Fourier Series and Boundary Value Problems (3rd ed.). McGraw-Hill.

3. Strauss, W. A. (2007). Partial Differential Equations: An Introduction (2nd ed.). John Wiley & Sons.

4. Haberman, R. (2012). Applied Partial Differential Equations with Fourier Series and Boundary Value Problems (5th ed.). Pearson.

5. Kreyszig, E. (2011). Advanced Engineering Mathematics (10th ed.). John Wiley & Sons.

6. Evans, L. C. (2010). Partial Differential Equations (2nd ed.). American Mathematical Society.

7. Logan, J. D. (2015). Applied Partial Differential Equations (3rd ed.). Springer.

8. Folland, G. B. (1992). Fourier Analysis and its Applications. American Mathematical Society.

9. Farlow, S. J. (1993). Partial Differential Equations for Scientists and Engineers. Courier Corporation (Dover Publications).

10. Myint-U, T., & Debnath, L. (2007). Linear Partial Differential Equations for Scientists and Engineers (4th ed.). Birkhäuser Boston.

11. Hagiwara, S. (2018). Convergence of the Fourier Series. https://doi.org/10.1007/978-0-387-79146-3_22