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Fourier Transform For Solving Wave and Schrödinger Equations

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تحويل فورييه لحل معادلات الموجة ومعادلات شرودنغر

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الملخص:
يهدف هذا البحث إلى دراسة المعادلات التفاضلية الجزئية، وتحديداً معادلة الموجة أحادية البعد وثنائية البعد، ومعادلة شرودنغر. وتعتمد الدراسة على طريقة فصل المتغيرات وتحويل فورييه كأدوات رئيسية للحصول على الحلول وتحليلها. كما تُقدم بعض المفاهيم الأساسية المتعلقة بمتسلسلات فورييه وتحويلات فورييه لتوضيح كيفية تمثيل الحلول بصيغة رياضية مبسطة تُساعد في فهم سلوك هذه المعادلات.

الكلمات الدالة: تحويل فورييه. المعادلات التفاضلية الجزئية. معادلة الموجة. معادلة شرودنغر.

Abstract
This research aims to study partial differential equations, specifically the one-dimensional and two-dimensional wave equation and the Schrödinger equation. The study relies on the method of separation of variables as well as the Fourier transform as main tools for obtaining and analyzing solutions. It also presents some basic concepts related to Fourier series and Fourier transforms in order to illustrate how solutions can be rep- resented in a simplified mathematical form that helps in understanding the behavior of these equations

Keywords: Fourier Transform .Partial Deferential Equations . Wave Equation . Schrödinger Equation.

Introduction

Fourier series continue to be one of the cornerstones of modern mathematical analysis, as it provides an effective way to instantiate the periodic cycle and analyze it into simple racial components. Since its inception, the latter has contributed to the development of much classical philosophy and engineering, especially in the study of wave phenomena and their propagation. The importance of the Fourier series is clearly evident in solving partial differential equations that describe physical viruses, as it leads to settling complex issues into simpler formulas that can be dealt with mathematically. The most prominent of these are the equations with the equations that describe the propagation of vibrations in different media, and the Schrödinger equation that fits the basis in quantum mechanics, as it describes the experience of the quantum state of the spheres. In the case of both, you want to be an immediate sequence of decomposition of solutions into independent frequency modularity, making the understanding of the physical structure integrated. With scientific progress, the applications of the Fourier series extended to broader fields such as time-frequency analysis and signal processing, and contributed to the emergence of modern techniques such as wavelet analysis. Therefore, the Fourier series remains a central tool in linking theoretical mathematics and physical applications, and a mainstay in the study of complex scientific models. we are going to solve the wave and telegraph equations on the full real line by Fourier transforming in the spatial variable.

2. Preliminaries:

We first recall some well-known definitions about Fourier series. These are used in **Theorem 1 (Bessel's Inequality) :** If$\int_{-\pi}^{\pi}|f(x)|^2 dx$ *is finite* ,than

$\sum_{n=-\infty}^{\infty}|C_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2 dx$ (1)

Bessel's inequality can be proved easily. In fact, we have

$0 \leq$

$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f(x)-\sum_{n=-N}^{N} C_n e^{inx}\right|^2 dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(f(x)-\sum_{m=-N}^{N} C_n e^{imx}\right)\left(\overline{f(x)}-\sum_{n=-N}^{N}\overline{C_n}e^{-inx}\right) dx$

Multiplying out the last integrand above,

$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f(x)-\sum_{n=-N}^{N} C_n e^{inx}\right|^2 dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2 dx - \sum_{n=-N}^{N}|C_n|^2$ (2)

Thus, for all N

$\sum_{n=-N}^{N}|C_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2 dx$ (3)

and Bessel's inequality (1) follows by letting$N \to \infty$,Bessel's inequality has a physical interpretation. If f has finite energy, in the sense that the right side of (1) is finite, then the sum of the moduli-squared of the Fourier coefficients is also finite, we shall see that the inequality in Eq. (1) is actually an equality, which says that the sum of the moduli-squared of the Fourier coefficients is precisely the same as the energy of . Because of Bessel's inequality, it follows that

$\lim_{|n|\to\infty} C_n = 0$ (4)

holds whenever$\int_{-\pi}^{\pi}|f(x)|^2 dx$ is finite. The Riemann-Lebesgue lemma says that Eq. (4) holds in the following more general case

Theorem 2 (Riemann-Lévesque Lemma) : If $\int_{-\pi}^{\pi}|f(x)|dx$ *is finite*, Eq. (4) holds.

Theorem 3 Suppose f has period 2π that $\int_{-\pi}^{\pi} f(x)dx$ is finite, and that f is Lipschitz at x_0Then the Fourier series for f converges to $f(x_0)$ at x_0 .

Theorem 4 Suppose that f has period 2π and is uniformly Lipschitz at all points.
Then the Fourier series for f converges uniformly to f .

Theorem 5 If f has period 2π and has bounded variation on $[0,2\pi]$ that the .Fourier series for f converges at all points. In fact, for all x-values,

$$\lim_{N\to\infty} S_N(x) = \frac{1}{2}[f(x+)+f(x-)]$$

Theorem 5 If f has period 2π and has bounded variation on $[0,2\pi]$ that the .Fourier series for f converges at all points. In fact, for all x-values,

$$\lim_{N\to\infty} S_N(x) = \frac{1}{2}[f(x+)+f(x-)]$$

3.The Wave Equation

If *u(x, t)* is the displacement from equilibrium of a string at position x and time t and if the string is undergoing small amplitude transverse vibrations, then we have seen that

$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t)$ (5)

for a constant c. We are now going to solve this equation by multiplying both sides by e^{-ikx} and integrating with respect to$x$. That is, we shall Fourier transform with respect to the spatial variable x. Denote the Fourier transform with respect to x, for each fixed t, of $u(x,t)$by

$$\hat{u}(k,t) = \int_{-\infty}^{\infty} u(x,t)e^{-ikx}dx$$

We have already seen (in property (D) in the notes "Fourier Transforms ") that the Fourier transform of the derivative *f′(x)* is,

$\int_{-\infty}^{\infty} f'(x)e^{-ikx}dx = ik\int_{-\infty}^{\infty} f(x)e^{-ikx}dx = ik\hat{f}(k)$ (6)

(by integration by parts with $u = e^{-ikx}, dv = f'(x)dx\,, du = -ike^{-ikx}dx\,, v = f(x)$and assuming that $f(x) \to 0 as x \to \pm\infty$). Applying this with $f(x) = \frac{\partial u(x,t)}{\partial x}$ and a second time with $f(x) = u(x,t)$,gives that the Fourier transform of $\frac{\partial^2 u}{\partial x^2}(x,t) is -$ first$t$–derivative,

$$\int_{-\infty}^{\infty} u_t(x,t)e^{-ikx}dx = \int_{-\infty}^{\infty} \lim_{h\to 0}\frac{u(x,t+h)-u(x,t)}{h}e^{-ikx}dx$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\int_{-\infty}^{\infty} u(x,t+h)e^{-ikx}dx - \int_{-\infty}^{\infty} u(x,t)e^{-ikx}dx\right)$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\hat{u}(k,t+h) - \hat{u}(k,t)\right)$$

$= \frac{\partial \hat{u}(k,t)}{\partial t}$ (7)

To get two t-derivatives, we just apply this twice (with u replaced by u_t the first time)

$$\int_{-\infty}^{\infty} \frac{\partial^2}{\partial t^2}u(x,t)e^{-ikx}dx = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} \frac{\partial}{\partial t}u(x,t)e^{-ikx}dx = \frac{\partial^2 \hat{u}}{\partial t^2}(k,t)$$

So applying the Fourier transform to both sides of (1) gives

$\frac{\partial^2 \hat{u}}{\partial t^2}(k,t) = -c^2k^2\hat{u}(k,t)$ (8)

This has not yet led to the solution for $u(x,t)$or $\hat{u}(k,t)$ but it has led to a considerable simplification. We now have, for each fixed k, a constant coefficient, homogeneous, second order ordinary differential equation for $\hat{u}(k,t)$, to emphasis that each k may now be treated independently, fix any k and write $\hat{u}(k,t) = U(t)$. The differential equation (8) now is $\frac{\partial^2 \hat{u}}{\partial t^2}(k,t) + c^2k^2\hat{u}(k,t) = 0$. From earlier courses, we know that this equation can be solved easily by trying $U(t) = e^{rt}$,Since $\frac{\partial^2 \hat{u}}{\partial t^2}(k,t) + c^2k^2\hat{u}(k,t) = (c^2k^2 + r^2)e^{rt} = 0$ if and only if $r = \pm ick$, the general solution to $\frac{\partial^2 \hat{u}}{\partial t^2}(k,t) + c^2k^2\hat{u}(k,t) = 0$, for any$k \neq 0$, is $U(t) = d_1e^{-ickt}$ $+d_2e^{ickt}$ .For $k = 0$, when the two values of $r = \pm ick$,are the same, the differential equation reduces to $\frac{\partial^2 \hat{u}}{\partial t^2}(k,t) = 0$. and has general solution is $U(t) = d_1 + d_2t$. We have to reject the d_2t solution (i.e. we have to require that $d_2 = 0$) on physical grounds small transverse oscillations certainly do not include amplitudes that grow to infinity at t goes to infinity. Recalling that $U(t) = \hat{u}(k,t)$we conclude that the general solution to (8) is

$\hat{u}(k,t) = \hat{F}(k)e^{-ickt} + \hat{G}(k)e^{ickt}$

We have renamed the arbitrary constantsd_1 and d_2 to $\hat{F}(k)$ and $\hat{G}(k)$, the arbitrary constants are certainly allowed to depend on k– viewed as an equation for an unknown function of t, (8) is a different equation for every different value of k. To recover $u(x,t)$ we just need to take the inverse Fourier transform

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(k,t)e^{ikx}dk$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} (\hat{F}(k)e^{-ickt} + \hat{G}(k)e^{ickt})e^{ikx}dk$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} (\hat{F}(k)e^{ik(x-ct)}\, dk + \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{G}(k)e^{ik(x+ct)}\, dk$$
$$= F(x-ct) + G(x+ct)$$

This is called the D'Alembert form of the solution of the wave equation.

4 Solving the wave equation by Fourier method

In this paper I will show how to solve an initial-boundary value problem for one dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2\frac{\partial^2 u}{\partial x^2}(x,t) \quad 0 < x < l\ ,\ t > 0, \qquad (9)$$

with the initial conditions

$$u(0,x) = f\ (x) \qquad ,0 < x < l$$
(10)
$$u(0,x) = g(x) \qquad ,0 < x < l$$

where f is the initial displacement and g is the initial velocity.

I start with the homogeneous boundary conditions of type I:

$$u(t,0) = 0,\ \ t > 0$$
(11)
$$u(t,l) = 0,\ \ t > 0$$

which physically means that I am studying the oscillations of a string of length l with *fixed* ends.

$$u(t,x) = T(t)X(x)$$

I find that

.

$$\frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda^2$$

and hence I have two ordinary differential equations

$$T'' + c^2\lambda^2 T = 0 \qquad (12)$$

for T, and

$$X'' + \lambda^2 X = 0 \qquad (13)$$

for X.

Using the boundary conditions (11) I conclude that equation (13) must be supplemented with the boundary conditions

$$X(0) = X(l) = 0 \qquad (14)$$

Problem (13)-(14) is a Sturm-Liouville eigenvalue problem, which we already solved several times. In particular, we know that there is an infinite series of eigenvalues

$$\lambda_k = \frac{k^2\pi^2}{l^2} \qquad ,k = 1,2,3,\ldots$$

and the corresponding eigenfunctions

$$X_k(x) = C_k sin\sqrt{\lambda_k}x = C_k sin\frac{k\pi x}{l} \quad k = 1,2,3,\ldots$$

moreover all the eigenfunctions are orthogonal on [0, l]. Here C_k are some arbitrary real constants .Since I know which lambdas I can use, I now can look at the solutions to (12). Since all my $\lambda_k > 0$I have the general solution

$$T_k(t) = A_k cos c\sqrt{\lambda_k} t + B_k sin c\sqrt{\lambda_k} t = A_k cos \frac{kc\pi t}{l} + B_k sin \frac{kc\pi t}{l}$$

and hence each function

$$u_k(t, x) = T_k(t) X_k(x) = (a_k cos \frac{kc\pi t}{l} + b_k sin \frac{kc\pi t}{l}) sin \frac{k\pi x}{l}$$

solves the wave equation(9), and satisfies the boundary conditions (11). Here $a_k = A_k C_k . b_k = C_k B_k$. Since my PDE is linear I can use the superposition principle to form my solution as

$$u(t, x) = \sum_{k=1}^{\infty} u_k(t, x)$$

my task is to determine a_k and b_k For this I will need the initial conditions. Note that using the first initial conditions implies

$$f(x) = \sum_{k=1}^{\infty} a_k sin \frac{k\pi x}{l}$$

which means that

$$a_k = \frac{2}{l} \int_0^l f(x) sin \frac{k\pi x}{l} dx$$

To use the second boundary condition I first differentiate my series and then plug in $t = 0$:

$$g(x) = \sum_{k=1}^{\infty} \frac{b_k c\pi k}{l} c sin \frac{\pi k x}{l}$$

which gives me

$$b_k = \frac{2}{\pi k c} \int_0^l g(x) sin \frac{k\pi x}{l} dx$$

Hence I found a formal solution to my original problem. since I must also check that all the series are convergent and can be differentiated twice in$t$ and$x$ to guarantee that what I found is a classical solution. Note that contrary to the heat equation, my series representations of solutions do not have quickly vanishing exponents and hence the question on differentiability is not as simple as before. Putting some additional smoothness requirements on the initial conditions can be used to conclude that my series are classical solutions.

$$A\cos\alpha t + B\sin\alpha t = \sqrt{A^2 + B^2}(\frac{A}{\sqrt{A^2 + B^2}} \cos\alpha t + \frac{B}{\sqrt{A^2 + B^2}} \sin\alpha t)$$

$$= \sqrt{A^2 + B^2}(\cos\emptyset\cos\alpha t + \sin\emptyset\sin\alpha t)$$

$$R\cos(\alpha t - \emptyset), \quad \emptyset = tan^{-1} \frac{B}{A} \ , R = \sqrt{A^2 + B^2},$$

I can rewrite the normal modes in the form

$$u(t, x) = R_k \cos\left(\frac{ck\pi t}{l} - \emptyset_k\right) sin \frac{k\pi x}{l}$$

Recall that Recall that *f* is periodic with period T if

$$f(t+T)=f(t)$$

for any t The (minimal) period is the smallest $T>0$in this formula. Clearly if f is T-periodic then

$$f(t+kT)=f(t) \qquad k\in Z$$

Moreover, if f is T-periodic then $f(at)$has period T/a . These basic facts imply that the normal modes are periodic functions with respect to variable t with the periods

$$T_k=\frac{2l}{ck}$$

because cosine is a 2π-periodic function. More importantly all normal modes have period $\frac{2l}{c}$ (not necessarily minimal), which allows me to conclude that the solution to the problem(9)-(11) is $\frac{2l}{c}$ -periodic:

$$T=2l\sqrt{\frac{\rho}{E}}$$

The angular frequencies are

$$w_k=\frac{2\pi}{T_k}=\frac{ck\pi}{l}$$

The normal modes are also called the harmonics. The first harmonic is the normal mode of the lowest frequency, which is called the fundamental frequency

$$w_1=\frac{c\pi}{l}$$

5-Separation of variables with several dimensions:

We now contrast the approach of Fourier transforming the equations with another standard technique. If we have a partial deferential equation for a function which depends on several variables, e.g. $u(x,y,z,t)$, then we can attempt to find a solution which is separable in the variables:

$$u(x,y,z,t)=X(x)Y(y)Z(z)T(t)$$

where $XYZT$ are some functions of their arguments, and we try to work out what these functions

Let us consider the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2}(x,y,t)=c^2\left(\frac{\partial^2 u}{\partial x^2}(x,y,t)+\frac{\partial^2 u}{\partial y^2}(x,y,t)\right) \qquad (15)$$

with the homogeneous boundary condition

$u(0,y,t)=0$,$u(a,y,t)=0$,

$u(x,0,t)=0$,$u(x,b,t)=0$,

with the initial conditions

$$u(x,y,0)=f(x,y)$$
$$u_t(x,y,0)=g(x,y) \qquad ,$$

where f is the initial displacement and g is the initial velocity.

we will look for separable solutions in the following form:

$$u(x,y,t)=X(x)Y(y)T(t)$$

I find that

$$\frac{T''}{T} = c^2(\frac{X''}{X} + \frac{Y''}{Y}) = -k^2$$

and hence I have three ordinary differential equations

$$T'' + c^2k^2T = 0$$

for T,

$$X'' + k_1^2X = 0$$

for X.and

$$Y'' + k_2^2Y = 0$$

for Y, the general solution

$$u(x,y,t) = (c_1cosk_1x + c_2sink_1x)(c_3cosk_2y + c_4sink_2y)$$
$$(c_5cosckt + c_6sinckt)$$

Using the boundary conditions

$u(0,y,t) = 0$,$u(a,y,t) = 0$,

$$0 = c_1Y(y)T(t)$$
$$c_1 = 0$$
$$x = a \quad ,u = 0$$
$$u = c_2\ sink_1aT(t)Y(y) = 0$$
$$c_2\ sink_1a = 0$$
$$sink_1a = 0 = sinm\pi$$
$$k_1a = m\pi$$
$$k_1 = \frac{m\pi}{a}$$

$u(x,0,t) = 0$,$u(x,b,t) = 0$,

$$c_2\ sin\frac{m\pi}{a}xc_3T(t) = 0$$
$$c_3 = 0$$

$$y = \mathrm{b}, u = 0$$

$$u = c_2\ sin\frac{m\pi}{a}xc_4\ sink_2bT(t) = 0$$
$$c_4\ sink_2b = 0$$

$$sink_2b = 0 = sinn\pi$$

$$k_2b = n\pi$$

$$k_2 = \frac{n\pi}{b}$$
$$u(x,y,t) = c_2sin\frac{m\pi}{a}xc_4sin\frac{n\pi}{b}y(c_5cosckt + c_6sinckt)$$
$$u(x,y,t) = sin\frac{m\pi}{a}xsin\frac{n\pi}{b}y(c_2c_4c_5cosckt + c_2c_4c_6sinckt)$$
$$c_2c_4c_5 = A_{mn}, c_2c_4c_6 = B_{mn}$$

$u(x,y,t) = sin\frac{m\pi}{a}xsin\frac{n\pi}{b}y(A_{mn}cosckt + B_{mn}sinckt)$ m,n=1.2…

Using the initial conditions

$u(x,y,0) = f\ (x,y)$

$$u(x,y,0) = sin\frac{m\pi}{a}xsin\frac{n\pi}{b}yA_{mn} = f\ (x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}sin\frac{m\pi}{a}xsin\frac{n\pi}{b}y$$

$$A_{mn} = \frac{2}{b}\frac{2}{a}\int_0^a\int_0^b f\ (x,y)sin\frac{m\pi}{a}xsin\frac{n\pi}{b}ydxdy$$

$u_t(x,y,0) = g(x,y\ )$

$u_t(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} sin\frac{m\pi}{a}xsin\frac{n\pi}{b}y\ \ kcB_{mn} = g(x,y\ )$

$$kcB_{mn} = \frac{2}{b}\frac{2}{a}\int_0^a\int_0^b g\ (x,y)sin\frac{m\pi}{a}xsin\frac{n\pi}{b}ydxdy$$

5-Fourier solution of the Schrödinger equation in 2D

Consider the time-dependent Schrödinger equation in 2D, for a particle trapped in a (zero) potential 2D square well with infinite potentials on walls at$x = 0, L, y = 0, L$

$\frac{-h^2}{2m}\nabla^2\psi(x,t) = ih\frac{\psi\partial(x,y)}{\partial t}$ (16)

For example, let us perform a FT with respect to $x, \nabla^2 \to -k, k = -k^2 = -\left(k_x^2 + k_y^2\right), so$

$$\frac{k^2h^2}{2m}\tilde{\psi}(k,t) = ih\frac{\tilde{\psi}\partial(k,y)}{\partial t}$$

We can integrate this with an integrating factor:

$$\tilde{\psi}(k,t) = \tilde{\psi}(k,0)e^{i\frac{-hk^2}{2m}t}$$

The time-dependence of the wave ise^{-iwt}, or, in terms of energy $E = hw, e^{-iEt/h}$

Where

$$E = \frac{k^2h^2}{2m}$$

If the particle is trapped in the box, then the wave function must be zero on the boundaries, so the wavelength in the $x$direction must be $\lambda_k = 2L, L, 2L/3\ , \dots i.e\ 2L/m\ .for\ q = 1.2.3\ \dots or$ wavenumbers of

$k_x = 2\pi/\ \lambda_k = q\pi/L$ similarly $k_y = r\pi/\ L\ for\ r = 1.2\ \dots$

So the wave function is a superposition of modes of the form

$$\psi(x,t) = Asin\frac{q\pi}{L}xsin\frac{r\pi}{L}ye^{-iEt/h}$$

for integers q, r. A is a normalization constant to ensure that $\int\int|\psi^2|dxdy = 1$.

Each mode has an energy

$$E = \frac{k^2h^2}{2m} = \frac{h^2\pi^2(q^2 + r^2)}{2mL^2}$$

For a square well, the energy levels are degenerate – different combinations of $q\ and\ r$give the same energy level.

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