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Local Functions and Composition With Euclidean Smooth Functions

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الملخص:

في [2] يتم تعريف الوظيفة المحلية على مجموعة غير فارغة والتركيب مع الدوال الملساء الإقليدية لمجموعة من الوظائف، وهو تعميم مجرد لمجموعة الوظائف في الفضاء الإقليدي [12]. توفر هذه الورقة الدوال المحلية والتركيب مع الدوال الإقليدية السلسة لمجموعة لا تحصى من الدوال. يتم تقديم نظريات وأمثلة مهمة تتعلق بالوظائف المحلية والتركيب مع الوظائف الإقليدية السلسة.

الكلمات الدالة: مفهوم الدوال، الطوبولوجيا الأولية، المشتقات الجزئية، الدوال الملساء.

Abstract

In [2] the local function on a nonempty set M and the composition with Euclidean smooth functions are defined for collection of functions C, which is an abstract generalization of the collection of C^∞ functions on the Euclidean space [12]. This paper provides local functions and composition with Euclidean smooth functions for a countable set of functions $C=\{f_1, f_2, f_3,..., f_n, n\in N\}$. Important theorems and examples concerning local functions and composition with Euclidean smooth functions are given.

Keywords: Functions concept, initial topology, partial derivatives, smooth functions.

Introduction

Throughout this paper, let $C=\{f_1, f_2, f_3,..., f_n, n\in N\}$ be a set of real-valued functions defined on a nonempty set M. A real function $f: M\to \mathbb{R}$, defined on a topological space (M,τ) is said to be a local C-function on M if, for any $p\in M$, there exist a neighborhood $U\in\tau$ of p and a function $g\in C=\{f_1, f_2, f_3,..., f_n, n\in N\}$ such that $f|U=g|U$ [12].

The set scC is defined by setting $f\in scC$ if and only if there exist $f^*_1, f_2^*,..., f_n^*\in C$, $n\in N$, and a function $\omega:\mathbb{R}^n\to\mathbb{R}$ of class C^∞ such that

$f=\omega\circ\left(f_1^*, f_2^*,..., f_n^*\right)$.

In other words[4]:

$$scC=\left\{\omega\circ\left(f_1^*, f_2^*,..., f_n^*\right): f_1^*, f_2^*,..., f_n^*\in C,\ \omega\in C^\infty(\mathbb{R}^n, \mathbb{R}),\ n\in N\right\}.$$

The paper is organized as follows. In **Section 2**, we present the basic definitions. This includes concepts in topology and analysis and some of theorems and examples are given. **Section 3**, provides the concept of local functions and some of theorems and examples concerning local functions are proved. Finally in **Section 4,** we studies the

2. Basic Definitions

Definition 2.1. [1] Let $\{(X_\lambda,\tau_\lambda)\}_{\lambda\in\Lambda}$ be a collection of topological spaces and let $C=\{f_1, f_2, f_3,..., f_n, n\in N\}$ be a collection of functions $f_\lambda: X\to X_\lambda$, where X is an arbitrary nonempty set, $\lambda=1,2,...,n, n\in N$. A topology on X, denoted by τ_C, is initial with respect to $C=\{f_1, f_2, f_3,..., f_n, n\in N\}$ if it has the following property: for any topological space Y, a function $g:(Y,\tau)\to(X,\tau_C)$ is continuous if and only if the composite $f_\lambda\circ g:(Y,\tau)\to(X_\lambda,\tau_\lambda)$ is continuous for each $\lambda=1,2,...,n,n\in N$.

We have the following theorem

Theorem 2.1. [1] Let τ_C be the initial topology on a nonempty set X with respect to $C=\{f_1, f_2, f_3,..., f_n, n\in N\}$. If τ is any topology on X such that each $f_\lambda:(X,\tau)\to(X_\lambda,\tau_\lambda)$ is continuous, then τ_C is weaker than τ , i.e., $\tau_C\subseteq\tau$.

Proof. Let $I_X:(X,\tau)\to(X,\tau_C)$ be the identity function. Since $f_\lambda=f_\lambda\circ I_X:(X,\tau)\to(X_\lambda,\tau_\lambda)$ is continuous for each $\lambda=1,2,...,n,n\in N$, then $I_X:(X,\tau)\to(X,\tau_C)$ is continuous. Consequently if $U\in\tau_C$, then $I_X^{-1}(U)=U\in\tau$. Hence $\tau_C\subseteq\tau$. ■

Example 2.1. The usual topology on $\mathbb{R}^n$ is the initial with respect to the projections $\pi_\lambda:\mathbb{R}^n\to\mathbb{R}$, where $\pi_\lambda(x_1,x_2,...,x_n)=x_\lambda,\ \lambda=1,2,3,...,n$. So a function $g=(g_1,g_2,...,g_n):Y\to\mathbb{R}^n$, where Y is a topological space, is continuous if and only if $\pi_\lambda\circ g=g_\lambda:Y\to\mathbb{R}$ is continuous.

Lemma 2.1. Let $C_1=\{f_1, f_2, f_3,..., f_n, n\in N\}, C_2=\{g_1, g_2, g_3,..., g_n, n\in N\}$ be two sets of real–valued functions on a nonempty set M. If $C_1\subseteq C_2$, then $\tau_{C_1}\subseteq\tau_{C_2}$.

Proof. A subbase of τ_{C_1} is $\beta_1=$ $\{f^{-1}(U): f\in C_1, U$ open in $\mathbb{R}\}$, a subbase of τ_{C_2} is

$\beta_2=$ $\{f^{-1}(U): f\in C_2, U$ open in $\mathbb{R}\}$

$=$ $\{f^{-1}(U): f\in C_1, U$ open in $\mathbb{R}\}\cup$

$\{f^{-1}(U): f\in C_2-C_1, U$ open in $\mathbb{R}\}$

$=$ β_1 $\cup\{f^{-1}(U): f\in C_2-C_1, U$ open in $\mathbb{R}\}$. Since $\beta_1\subseteq\beta_2$, then $\tau_{C_1}\subseteq\tau_{C_2}$. ■

Definition 2.2. [9] Let G be an open subset of $\mathbb{R}^n$. A function $f: G\to$ $\mathbb{R}^k$ is called infinitely differentiable, or of class C^∞ provided all partial derivatives of f, of all orders, exist and are continuous on G.

Let $C^\infty(G, \mathbb{R}^k)$ denotes the set of all functions $f: G\to$ $\mathbb{R}^k$ of class C^∞. Or more generally (see, for instance, [9]).

Definition 2.3. Let G be an open subset of $\mathbb{R}^n$, let r be a positive integer. A function $f: G\to$ $\mathbb{R}^k$ is said to be of class C^r if all its partial derivatives up to the order r exist and are continuous on G. The set of all C^r functions $f: G\to$ $\mathbb{R}^k$ is denoted by $C^r(G, \mathbb{R}^k)$. Thus $f\in C^\infty(G, \mathbb{R}^k)$ if and only if $f\in C^r(G, \mathbb{R}^k)$ for $r=0,1,2, \cdots$, where $C^0(G, \mathbb{R}^k)$ is the set of all continuous functions on G with values in $\mathbb{R}^k$.

3. Local Functions

Let (M,τ) be a topological space with a topology τ, and $C=\{f_1, f_2, f_3,..., f_n, n\in N\}$ a set of real–valued functions defined on M. As in [3, 4, 6, 8, 12 and 14], local functions are defined as following :

Definition 3.1. A real function $f: M\to\mathbb{R}$, defined on a topological space (M,τ) is said to be a local C–function on M if, for any $p\in M$, there exist a neighborhood $U\in\tau$ of p and a function $g\in C=\{f_1, f_2, f_3,..., f_n, n\in N\}$ such that $f|U=g|U$. The set of all local C–functions on M will be denoted by $C_{(M,\tau)}$ or, simply, C_M. Another way to define a local C–function is the following [12]:

A function f defined on a topological space M is a local C–function provided there exists an open covering $\mathcal{U}$ of the space M, such that for every set $U \in \mathcal{U}$ there exists a function $g_U \in C = \{f_1, f_2, f_3, ..., f_n, n \in N\}$ with $f|U = g_U|U$.

The following lemma is stated without proof in [12].

Lemma 3.1. Let $C = \{f_1, f_2, f_3, ..., f_n, n \in N\}$ be a set of real–valued functions over a topological space (M, τ). Then every $f \in C$ is a local C–function on M, i.e., $C \subseteq C_M$.

Proof. Let $f \in C$. Let $p \in M$. Take $g = f$ and $U = M$, then $f|U = g|U$. It follows that $f \in C_M$. Hence $C \subseteq C_M$. ■

The following definition is given in [20] and adopted by [3, 4, 11 and 14].

Definition 3.2. A set of real–valued functions $C = \{f_1, f_2, f_3, ..., f_n, n \in N\}$ on a topological space M is said to be closed with respect to localization if $C = C_M$. To prove that $C = C_M$, it is enough to show that $C_M \subseteq C$.

Lemma 3.2. Let $C = \{f_1, f_2, f_3, ..., f_n, n \in N\}$ be a set of real–valued functions over a topological space (M, τ). Then C is closed with respect to localization if and only if $C_M \subseteq C$.

Proof. If C is closed with respect to localization, i.e., $C = C_M$, then $C_M \subseteq C$. Conversely, if $C_M \subseteq C$, then by Lemma 3.1 we have $C \subseteq C_M$. Therefore $C = C_M$, so C is closed with respect to localization. ■

Theorem 3.1. Let $C = \{f_1, f_2, f_3, ..., f_n, n \in N\}$ be a set of real–valued functions on a topological space M. Then $(C_M)_M = C_M$.

Proof. Let $D = C_M$. Then we want to show that $D_M = D$. By Lemma 3.2, it is enough to show that $D_M \subseteq D$. Now, if $f \in D_M$, then for each $p \in M$ there exist a neighborhood $U \in \tau$ of p and a function $g \in D$ such that $f|U = g|U$. Since $g \in D = C_M$, then there exist a neighborhood V of p and a function $h \in C$ with $g|V = h|V$. Since $W = U \cap V$ is a neighborhood of $p \in M$, $h \in C$, and $f|W = h|W$, then $f \in C_M = D$. Hence

$D_M \subseteq D$. Therefore $D_M = D$, by Definition 3.2, we have $D = C_M$ is closed with respect to localization. ■

Example 3.1. Let $C = \{f_1, f_2, f_3, ..., f_n, n \in N\}$ be the set of all continuous real−valued functions on a topological space M. Then C is closed with respect to localization, i.e., $C_M = C$.

Proof. Let $f \in C_M$. Then there exists an open cover $\mathcal{U}$ of M such that for each $U \in \mathcal{U}$ there exists a function $g_U \in C$ with $f|U = g_U|U$. Now, let V be an open set in $\mathbb{R}$. Then

$$\begin{aligned} f^{-1}(V) &= M \cap f^{-1}(V) \\ &= \bigcup_{U \in \mathcal{U}} U \cap f^{-1}(V) \\ &= \bigcup_{U \in \mathcal{U}} U \cap g_U^{-1}(V) \end{aligned}$$

which is open, because g_U is continuous and U is open. Then f is continuous. Hence $f \in C$. So $C_M \subseteq C$, by Lemma 3.2, C is closed with respect to localization. ■

The following lemma is in order.

Lemma 3.3. Let $C = \{f_1, f_2, f_3, ..., f_n, n \in N\}$ be a set of real−valued functions on a nonempty set M. Let τ_1 and τ_2 be two topologies on M. If $\tau_1 \subseteq \tau_2$, then $C_{(M,\tau_1)} \subseteq C_{(M,\tau_2)}$.

Proof. Let $f \in C_{(M,\tau_1)}$. Then for each $p \in M$, there exist a neighborhood $U_p \in \tau_1$ and a function $g \in C$ such that $f|U_p = g|U_p$. Since $\tau_1 \subseteq \tau_2$, then $U_p \in \tau_2$ and $f \in C_{(M,\tau_2)}$. Consequently $C_{(M,\tau_1)} \subseteq C_{(M,\tau_2)}$. ■

Lemma 3.4. Let $C = \{f_1, f_2, f_3, ..., f_n, n \in N\}$ and $D = \{g_1, g_2, g_3, ..., g_n, n \in N\}$ be two sets of real−valued functions on a nonempty set M and let τ be a topology on M. If $C \subseteq D$, then $C_M \subseteq D_M$.

Proof. Let $f \in C_M$. Then for each $p \in M$, there exist a neighborhood $U_p \in \tau$ and a function $g \in C$ such that $f|U_p = g|U_p$. Since $C \subseteq D$, then $g \in D$. Hence $f \in D_M$. Thus $C_M \subseteq D_M$. ■

Lemma 3.5. Let $C=\{f_1, f_2, f_3,..., f_n, n\in N\}$ and $D=\{g_1, g_2, g_3,..., g_n, n\in N\}$ be two sets of real-valued functions on a set M and let τ_1, τ_2 be two topologies on M. If $C\subseteq D$ and $\tau_1\subseteq\tau_2$, then $C_{(M,\tau_1)}\subseteq D_{(M,\tau_2)}$.

Proof. From Lemma 3.4, we have $C_{(M,\tau_1)}\subseteq D_{(M,\tau_1)}$. (1)

By Lemma 3.3, we have $D_{(M,\tau_1)}\subseteq D_{(M,\tau_2)}$. (2)

From (1) and (2), we have $C_{(M,\tau_1)}\subseteq D_{(M,\tau_2)}$. ■

4. Composition with Euclidean Smooth Functions.

Let $C=\{f_1, f_2, f_3,..., f_n, n\in N\}$ be a set of real-valued functions defined on a nonempty set M. As in [8, 11 and 14], the set scC is defined by setting $f\in scC$ if and only if there exist $f^*_1, f_2^*,..., f_n^*\in C$, $n\in N$, and a function $\omega:\mathbb{R}^n\to\mathbb{R}$ of class C^∞ such that $f=\omega\circ\left(f_1^*, f_2^*,..., f_n^*\right)$.

In other words [4, 5, 6, 10, 11 and 14]:

$scC=\left\{\omega\circ\left(f_1^*, f_2^*,..., f_n^*\right): f_1^*, f_2^*,..., f_n^*\in C, \omega\in C^\infty(\mathbb{R}^n, \mathbb{R}), n\in N\right\}$.

Lemma 4.1. Let $C=\{f_1, f_2, f_3,..., f_n, n\in N\}$ be a set of real-valued functions defined on a nonempty set M. Then $C\subseteq scC$.

Proof. Let $f\in C$ and let $I_{\mathbb{R}}:\mathbb{R}\to\mathbb{R}$ be the identity function. Since $I_{\mathbb{R}}\in C^\infty(\mathbb{R}, \mathbb{R})$, then $I_{\mathbb{R}}\circ f\in scC$. Since $f=I_{\mathbb{R}}\circ f$, then $f\in scC$. Hence $C\subseteq scC$. ■

Corollary 4.1. Let $C=\{f_1, f_2, f_3,..., f_n, n\in N\}$ be a set of real-valued functions on a nonempty set M. Then $scC=C$ if and only if $scC\subseteq C$.

Proof. If $scC=C$, then $scC\subseteq C$. Conversely, if $scC\subseteq C$, then by the preceding lemma we have $C\subseteq scC$. Consequently $scC=C$. ■

The following definition is given in [4, 11, 12 and 14].

Definition 4.1. Let $C=\{f_1, f_2, f_3,..., f_n, n\in N\}$ be a set of real-valued functions over a nonempty set M. If $scC=C$, then C is said to be closed with respect to composition with Euclidean smooth functions.

Lemma 4.2. Let C_1, C_2 be two sets of real-valued functions on a nonempty set M. If $C_1\subseteq C_2$, then $scC_1\subseteq scC_2$.

Proof. Let $f \in scC_1$. Then $f = \omega \circ (f_1, f_2, ..., f_n)$ for some $f_1, f_2, ..., f_n \in C_1$, $n \in N$, and some $\omega \in C^{\infty}(\mathbb{R}^n, \mathbb{R})$. Since $C_1 \subseteq C_2$, then $f_1, f_2, ..., f_n \in C_2$ and $f \in scC_2$. Hence $scC_1 \subseteq scC_2$. ■

Example 4.1. Let $C = \{f_1, f_2, f_3, ..., f_n, n \in N\}$ be the set of all continuous real−valued functions on a topological space M. Then $scC \subseteq C$.

proof. Let $f \in scC$. Then there exist $f_1, f_2, ..., f_n \in C$, $n \in N$, and a function $\omega \in C^{\infty}(\mathbb{R}^n, \mathbb{R})$ such that $f = \omega \circ (f_1, f_2, ..., f_n)$. Then f is continuous because ω is continuous, $(f_1, f_2, ..., f_n)$ is a continuous function and the composition of two continuous functions is a continuous function. This means that $f \in C$. Hence $scC \subseteq C$. ■

Theorem 4.1. Let $C_0 = \{f_1, f_2, f_3, ..., f_n, n \in N\}$ be a set of real−valued functions defined on a nonempty set M. Then the initial topology with respect to C_0 and scC_0 coincide, i.e., $\tau_{C_0} = \tau_{scC_0}$.

proof. Let $f \in C_0$. By Lemma 4.1, $C_0 \subseteq scC_0$. Then $f \in scC_0$ and $f : (M, \tau_{scC_0}) \to \mathbb{R}$ is continuous for each $f \in C_0$. But τ_{C_0} is the weakest topology such that each $f \in C_0$ is continuous. Hence $\tau_{C_0} \subseteq \tau_{scC_0}$. (3)

On the other hand, let $g \in scC_0$. Then $g = \omega \circ (f^*_1, f_2^*, ..., f_n^*)$ for some $f_1^*, f_2^*, ..., f_n^* \in C_0$, where $n \in N$, and some $\omega \in C^{\infty}(\mathbb{R}^n, \mathbb{R})$. Since the topology of $\mathbb{R}^n$ is the initial with respect to the projections $\pi_i : \mathbb{R}^n \to \mathbb{R}$, where $\pi_i(x_1, x_2, ..., x_n) = x_i$, and $\pi_i \circ (f_1^*, f_2^*, ..., f_n^*) = f_i : (M, \tau_{C_0}) \to \mathbb{R}$ is continuous, then $(f_1^*, f_2^*, ..., f_n^*) : (M, \tau_{C_0}) \to \mathbb{R}^n$ is continuous. So the composition $g = \omega \circ (f_1^*, f_2^*, ..., f_n^*) : (M, \tau_{C_0}) \to \mathbb{R}$ of two continuous functions is continuous. Hence $g : (M, \tau_{C_0}) \to \mathbb{R}$ is continuous for all $g \in scC_0$. But τ_{scC_0} is the weakest topology on M such that each $g \in scC_0$ is continuous. It follows that $\tau_{scC_0} \subseteq \tau_{C_0}$. (4)

From (3) and (4), we have $\tau_{C_0} = \tau_{scC_0}$. ■

In general, by Lemma 3.1, $C \subseteq C_M$ nevertheless the following theorem says that $\tau_C = \tau_{C_M}$.

Theorem 4.2. Let $C=\{f_1, f_2, f_3,..., f_n, n\in N\}$ be a set of real−valued functions defined on a nonempty set M. Then $\tau_C=\tau_{C_M}$, where C_M is the set of all local C−functions with respect to the topology τ_C on M.

Proof. Let $f\in C_M$. Then for each $p\in M$, there exist a neighborhood $U_p\in\tau_C$ of p and a function $g_p\in C$ such that $f|U_p=g_p|U_p$. Since $g_p:(M,\tau_C)\to\mathbb{R}$ is continuous, then $f:(M,\tau_C)\to\mathbb{R}$ is continuous at p. Since $p\in M$ is arbitrary, then $f:(M,\tau_C)\to\mathbb{R}$ is continuous. But this means that $f:(M,\tau_C)\to\mathbb{R}$ is continuous for any $f\in C_M$. Since τ_{C_M} is the weakest topology on M such that each $f\in C_M$ is continuous, then $\tau_{C_M}\subseteq\tau_C$. (5)

Conversely, suppose that $f\in C$. Since, by Lemma 3.1, we have $C\subseteq C_M$, then $f\in C_M$. Therefore $f:(M,\tau_{C_M})\to\mathbb{R}$ is continuous for all $f\in C$. But τ_C is the weakest topology on M such that f is continuous for all $f\in C$. Consequently $\tau_C\subseteq\tau_{C_M}$. (6)

From (5) and (6), we have $\tau_C=\tau_{C_M}$. ■

Theorem 4.3. Let $C=\{f_1, f_2, f_3,..., f_n, n\in N\}$ be a set of real−valued functions defined on a nonempty set M and consider M with the topology τ_C. Then $\tau_C=\tau_{scC}=\tau_{C_M}=\tau_{(scC)_M}=\tau_{scC_M}$.

Proof. From Theorem 4.1 we have $\tau_C=\tau_{scC}$ (7)

And $\tau_{C_M}=\tau_{scC_M}$. (8)

And by Theorem 4.2, we have $\tau_C=\tau_{C_M}$ (9)

And $\tau_{scC}=\tau_{(scC)_M}$. (10)

From (7), (8), (9) and (10), we have

$\tau_C=\tau_{scC}=\tau_{C_M}=\tau_{(scC)_M}=\tau_{scC_M}$. ■

Theorem 4.4. Let $C_0=\{f_1, f_2, f_3,..., f_n, n\in N\}$ be a set of real−valued functions defined on a nonempty set M. If $C=(scC_0)_{(M,\tau_{C_0})}$ with respect to the topology τ_{C_0}, then $C_{(M,\tau_C)}\subseteq C$ and C is closed with respect to Localization.

Proof. Let $f \in C_{(M,\tau_C)}$. That is, for each $p \in M$, there exist a neighborhood $U_P \in \tau_C$ of p with $p \in U_p \in \tau_C$ and a function $g_p \in C$ such that $f|U_p = g_p|U_p$. (11)

Since $g_p \in C = (scC_0)_{(M,\tau_{C_0})}$, then there exist a neighborhood $V_p \in \tau_{C_0}$ of p and a function $h_p \in scC_0$ with $g_p|V_p = h_p|V_p$. (12)

Since $h_p \in scC_0$, then $h_p = \omega \circ \left(f_1^*, f_2^*, ..., f_n^* \right)$ (13)

for some finite sequence $f_1^*, f_2^*, ..., f_n^* \in C_0$, and some $\omega \in C^\infty(\mathbb{R}^n, \mathbb{R})$. From (11), (12) and (13), we have $f|U_p \cap V_p = \omega \circ \left(f_1^*, f_2^*, ..., f_n^* \right)|U_p \cap V_p$.

Since $\tau_{C_0} = \tau_C$, then $U_p \cap V_p \in \tau_{C_0}$ is an open neighborhood of p. This implies that $f \in (scC_0)_{(M,\tau_{C_0})} = C$. Hence $C_{(M,\tau_C)} \subseteq C$. By Lemma 3.2, C is closed with respect to Localization. ■

Theorem 4.5. Let $C_0 = \{f_1, f_2, f_3, ..., f_n, n \in N\}$ be a set of real−valued functions on a nonempty set M and consider M with the topology τ_{C_0}. Then the set of all local scC_0 – functions on M, $C = (scC_0)_M$, is closed with respect to composition with smooth functions, i.e., $scC = C$.

Proof. Let us observe that $\tau_{C_0} = \tau_{scC_0} = \tau_C$. Let $f \in scC$.

We want to show that $f \in C$. Since $f \in scC$, then there exist $f_1, f_2, ..., f_n \in C$ and $\omega \in C^\infty(\mathbb{R}^n, \mathbb{R})$ such that $f = \omega \circ \left(f_1^*, f_2^*, ..., f_n^* \right)$. (14)

Since $f_1^*, f_2^*, ..., f_n^* \in C = (scC_0)_M$, then for each $p \in M$ there exist neighborhoods $U_1, U_2, ..., U_n$ of p and functions $g_1, g_2, ..., g_n \in scC_0$ such that $f_1|U_1 = g_1|U_1, ..., f_n|U_n = g_n|U_n$. Let $U_p = U_1 \cap U_2 \cap ... \cap U_n$. Then

$f_1|U_p = g_1|U_p, ..., f_n|U_p = g_n|U_p$. (15)

Since $g_1, g_2, ..., g_n \in scC_0$, then

$$\left.\begin{array}{l} g_1 = \omega_1 \circ \left(g_1^1, ..., g_1^{k_1} \right) \\ \vdots \\ g_n = \omega_n \circ \left(g_n^1, ..., g_n^{k_n} \right) \end{array}\right\} \quad (16)$$

for some $\omega_1 \in C^\infty(\mathbb{R}^{k_1}, \mathbb{R})$, $\cdots$, $\omega_n \in C^\infty(\mathbb{R}^{k_n}, \mathbb{R})$ and $g_1^1,...,g_1^{k_1},...,g_n^1,...,g_n^{k_n} \in C_0$. From (14), (15) and (16), it follows that $f\left|U_p = \omega \circ \left(\omega_1 \circ \left(g_1^1,...,g_1^{k_1}\right),...,\omega_n \circ \left(g_n^1,...,g_n^{k_n}\right)\right)\right|U_p$. Thus $f\left|U_p = \omega \circ (\theta_1, \theta_2 ..., \theta_n) \circ \left(g_1^1,...,g_1^{k_1},...,g_n^1,...,g_n^{k_n}\right)\right|U_p$ where $\theta_1, \theta_2, ..., \theta_n$ are functions in $C^\infty(\mathbb{R}^{k_1+...+k_n}, \mathbb{R})$ defined by

$$\theta_1\left(x_1^1,...,x_1^{k_1},...,x_n^1,...,x_n^{k_n}\right) = \omega_1\left(x_1^1,...,x_1^{k_1}\right)$$
$$\vdots$$
$$\theta_n\left(x_1^1,...,x_1^{k_1},...,x_n^1,...,x_n^{k_n}\right) = \omega_n\left(x_n^1,...,x_n^{k_n}\right).$$

Let $g_p = \omega \circ (\theta_1,...,\theta_n) \circ \left(g_1^1,...,g_1^{k_1},...,g_n^1,...,g_n^{k_n}\right)$. Then, for each $p \in M$, there exist a neighborhood U_p of p and a function $g_p \in scC_0$ such that $f\left|U_p = g_p\right|U_p$. Consequently $f \in (scC_0)_M = C$. This is true for all $f \in scC$. Hence $scC \subseteq C$. This completes the proof.

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