Local Functions and Composition With Euclidean Smooth Functions

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Abstract

In [2] the local function on a nonempty set $M$ and the composition with Euclidean smooth functions are defined for collection of functions $C$, which is an abstract generalization of the collection of $C^\infty$ functions on the Euclidean space [12]. This paper provides local functions and composition with Euclidean smooth functions for a countable set of functions $C = \{f_1, f_2, f_3, \ldots, f_n, n \in N\}$. Important theorems and examples concerning local functions and composition with Euclidean smooth functions are given.

Keywords: Functions concept, initial topology, partial derivatives, smooth functions.

Introduction

Throughout this paper, let $C = \{f_1, f_2, f_3, \ldots, f_n, n \in N\}$ be a set of real-valued functions defined on a nonempty set $M$. A real function $f : M \to \mathbb{R}$, defined on a topological space $(M, \tau)$ is said to be a local $C$-function on $M$ if, for any $p \in M$, there exist a neighborhood $U \in \tau$ of $p$ and a function $g \in C = \{f_1, f_2, f_3, \ldots, f_n, n \in N\}$ such that $f|_U = g|_U$ [12].

The set $scC$ is defined by setting $f \in scC$ if and only if there exist $f_1^*, f_2^*, \ldots, f_n^* \in C$, $n \in N$, and a function $\omega : \mathbb{R}^n \to \mathbb{R}$ of class $C^\infty$ such that
\[ f = \omega \circ (f_1^*, f_2^*, \ldots, f_n^*) \].

In other words[4]:

\[ \text{scC} = \{ \omega \circ (f_1^*, f_2^*, \ldots, f_n^*): f_1^*, f_2^*, \ldots, f_n^* \in C, \omega \in C^\infty (\mathbb{R}^n, \mathbb{R}), n \in \mathbb{N} \} \].

The paper is organized as follows. In Section 2, we present the basic definitions. This includes concepts in topology and analysis and some of theorems and examples are given. Section 3, provides the concept of local functions and some of theorems and examples concerning local functions are proved. Finally in Section 4, we studies the

2. Basic Definitions

**Definition 2.1.** [1] Let \( \{(X_\lambda, \tau_\lambda)\}_{\lambda \in \Lambda} \) be a collection of topological spaces and let \( C = \{f_1, f_2, f_3, \ldots, f_n, n \in \mathbb{N}\} \) be a collection of functions \( f_\lambda: X \to X_\lambda \), where \( X \) is an arbitrary nonempty set, \( \lambda = 1, 2, \ldots, n, n \in \mathbb{N} \). A topology on \( X \), denoted by \( \tau_c \), is initial with respect to \( C = \{f_1, f_2, f_3, \ldots, f_n, n \in \mathbb{N}\} \) if it has the following property: for any topological space \( Y \), a function \( g: (Y, \tau) \to (X, \tau_c) \) is continuous if and only if the composite \( f_\lambda \circ g: (Y, \tau) \to (X_\lambda, \tau_\lambda) \) is continuous for each \( \lambda = 1, 2, \ldots, n, n \in \mathbb{N} \).

We have the following theorem

**Theorem 2.1.** [1] Let \( \tau_c \) be the initial topology on a nonempty set \( X \) with respect to \( C = \{f_1, f_2, f_3, \ldots, f_n, n \in \mathbb{N}\} \). If \( \tau \) is any topology on \( X \) such that each \( f_\lambda: (X, \tau) \to (X_\lambda, \tau_\lambda) \) is continuous, then \( \tau_c \) is weaker than \( \tau \), i.e., \( \tau_c \subseteq \tau \).

**Proof.** Let \( I_X: (X, \tau) \to (X, \tau_c) \) be the identity function. Since \( f_\lambda = f_\lambda \circ I_X: (X, \tau) \to (X_\lambda, \tau_\lambda) \) is continuous for each \( \lambda = 1, 2, \ldots, n, n \in \mathbb{N} \), then \( I_X: (X, \tau) \to (X, \tau_c) \) is continuous. Consequently if \( U \in \tau_c \), then \( I_X^{-1}(U) = U \in \tau \). Hence \( \tau_c \subseteq \tau \).

**Example 2.1.** The usual topology on \( \mathbb{R}^n \) is the initial with respect to the projections \( \pi_\lambda: \mathbb{R}^n \to \mathbb{R}, \quad \pi_\lambda(x_1, x_2, \ldots, x_n) = x_\lambda, \lambda = 1, 2, 3, \ldots, n \). So a function \( g = (g_1, g_2, \ldots, g_n): Y \to \mathbb{R}^n \), where \( Y \) is a topological space, is continuous if and only if \( \pi_\lambda \circ g = g_\lambda: Y \to \mathbb{R} \) is continuous.
Lemma 2.1. Let \( C_1 = \{f_1, f_2, f_3, \ldots, f_n, n \in N\}, C_2 = \{g_1, g_2, g_3, \ldots, g_n, n \in N\} \) be two sets of real-valued functions on a nonempty set \( M \). If \( C_1 \subseteq C_2 \), then \( \tau_{C_1} \subseteq \tau_{C_2} \).

Proof. A subbase of \( \tau_{C_1} \) is \( \beta_1 = \{f^{-1}(U) : f \in C_1, U \text{ open in } \mathbb{R}\} \), a subbase of \( \tau_{C_2} \) is \( \beta_2 = \{f^{-1}(U) : f \in C_2, U \text{ open in } \mathbb{R}\} \)

\[
= \{f^{-1}(U) : f \in C_1, U \text{ open in } \mathbb{R}\} \cup \{f^{-1}(U) : f \in C_2 - C_1, U \text{ open in } \mathbb{R}\}
\]

Since \( \beta_1 \subseteq \beta_2 \), then \( \tau_{C_1} \subseteq \tau_{C_2} \).

Definition 2.2. [9] Let \( G \) be an open subset of \( \mathbb{R}^n \). A function \( f : G \to \mathbb{R}^k \) is called infinitely differentiable, or of class \( C^\infty \), provided all partial derivatives of \( f \), of all orders, exist and are continuous on \( G \).

Let \( C^\infty(G, \mathbb{R}^k) \) denotes the set of all functions \( f : G \to \mathbb{R}^k \) of class \( C^\infty \). Or more generally (see, for instance, [9]).

Definition 2.3. Let \( G \) be an open subset of \( \mathbb{R}^n \), let \( r \) be a positive integer. A function \( f : G \to \mathbb{R}^k \) is said to be of class \( C^r \) if all its partial derivatives up to the order \( r \) exist and are continuous on \( G \). The set of all \( C^r \) functions \( f : G \to \mathbb{R}^k \) is denoted by \( C^r(G, \mathbb{R}^k) \). Thus \( f \in C^\infty(G, \mathbb{R}^k) \) if and only if \( f \in C^r(G, \mathbb{R}^k) \) for \( r = 0, 1, 2, \ldots \), where \( C^0(G, \mathbb{R}^k) \) is the set of all continuous functions on \( G \) with values in \( \mathbb{R}^k \).

3. Local Functions

Let \( (M, \tau) \) be a topological space with a topology \( \tau \), and \( C = \{f_1, f_2, f_3, \ldots, f_n, n \in N\} \) a set of real–valued functions defined on \( M \). As in [3, 4, 6, 8, 12 and 14], local functions are defined as following:

Definition 3.1. A real function \( f : M \to \mathbb{R} \), defined on a topological space \( (M, \tau) \) is said to be a local \( C \)–function on \( M \) if, for any \( p \in M \), there exist a neighborhood \( U \in \tau \) of \( p \) and a function \( g \in C = \{f_1, f_2, f_3, \ldots, f_n, n \in N\} \) such that \( f|U = g|U \). The set of all
local \( C \)-functions on \( M \) will be denoted by \( C_{(M,\tau)} \) or, simply, \( C_M \). Another way to define a local \( C \)-function is the following [12]:

A function \( f \) defined on a topological space \( M \) is a local \( C \)-function provided there exists an open covering \( \mathcal{U} \) of the space \( M \), such that for every set \( U \in \mathcal{U} \) there exists a function \( g_U \in C = \{ f_1, f_2, f_3, \ldots, f_n, n \in N \} \) with \( f|_U = g_U|_U \).

The following lemma is stated without proof in [12].

**Lemma 3.1.** Let \( C = \{ f_1, f_2, f_3, \ldots, f_n, n \in N \} \) be a set of real-valued functions over a topological space \((M, \tau)\). Then every \( f \in C \) is a local \( C \)-function on \( M \), i.e., \( C \subseteq C_M \).

**Proof.** Let \( f \in C \). Let \( p \in M \). Take \( g = f \) and \( U = M \), then \( f|_U = g|_U \). It follows that \( f \in C_M \). Hence \( C \subseteq C_M \). ■

The following definition is given in [20] and adopted by [3, 4, 11 and 14].

**Definition 3.2.** A set of real-valued functions \( C = \{ f_1, f_2, f_3, \ldots, f_n, n \in N \} \) on a topological space \((M, \tau)\) is said to be closed with respect to localization if \( C = C_M \). To prove that \( C = C_M \), it is enough to show that \( C_M \subseteq C \).

**Lemma 3.2.** Let \( C = \{ f_1, f_2, f_3, \ldots, f_n, n \in N \} \) be a set of real-valued functions over a topological space \((M, \tau)\). Then \( C \) is closed with respect to localization if and only if \( C \subseteq C_M \).

**Proof.** If \( C \) is closed with respect to localization, i.e., \( C = C_M \), then \( C_M \subseteq C \). Conversely, if \( C \subseteq C_M \), then by Lemma 3.1 we have \( C \subseteq C_M \). Therefore \( C = C_M \), so \( C \) is closed with respect to localization. ■

**Theorem 3.1.** Let \( C = \{ f_1, f_2, f_3, \ldots, f_n, n \in N \} \) be a set of real-valued functions on a topological space \( M \). Then \( (C_M)_M = C_M \).

**Proof.** Let \( D = C_M \). Then we want to show that \( D_M = D \). By Lemma 3.2, it is enough to show that \( D_M \subseteq D \). Now, if \( f \in D_M \), then for each \( p \in M \) there exist a neighborhood \( U \in \tau \) of \( p \) and a function \( g \in D \) such that \( f|_U = g|_U \). Since \( g \in D = C_M \), then there exist a neighborhood \( V \) of \( p \) and a function \( h \in C \) with \( g|V = h|V \). Since \( W = U \cap V \) is a neighborhood of \( p \), \( h \in C \), and \( f|W = h|W \), then \( f \in C_M = D \). Hence
\[ D_M \subseteq D. \] Therefore \( D_M = D \), by Definition 3.2, we have \( D = C_M \) is closed with respect to localization. ■

**Example 3.1.** Let \( C = \{f_1, f_2, f_3, \ldots, f_n, n \in \mathbb{N}\} \) be the set of all continuous real-valued functions on a topological space \( M \). Then \( C \) is closed with respect to localization, i.e., \( C_M = C \).

**Proof.** Let \( f \in C_M \). Then there exists an open cover \( \mathcal{U} \) of \( M \) such that for each \( U \in \mathcal{U} \) there exists a function \( g_U \in C \) with \( f \big| U = g_U \big| U \). Now, let \( V \) be an open set in \( \mathbb{R} \).

Then
\[
\left( f^{-1}(V) \right) = \bigcup_{U \in \mathcal{U}} U \bigcap f^{-1}(V) = \bigcup_{U \in \mathcal{U}} U \bigcap \bigcup_{U \in \mathcal{U}} g_U^{-1}(V)
\]
which is open, because \( g_U \) is continuous and \( U \) is open. Then \( f \) is continuous. Hence \( f \in C \). So \( C_M \subseteq C \), by Lemma 3.2, \( C \) is closed with respect to localization. ■

The following lemma is in order.

**Lemma 3.3.** Let \( C = \{f_1, f_2, f_3, \ldots, f_n, n \in \mathbb{N}\} \) be a set of real-valued functions on a nonempty set \( M \). Let \( \tau_1 \) and \( \tau_2 \) be two topologies on \( M \). If \( \tau_1 \subseteq \tau_2 \), then \( C_{(M, \tau_1)} \subseteq C_{(M, \tau_2)} \).

**Proof.** Let \( f \in C_{(M, \tau_1)} \). Then for each \( p \in M \), there exist a neighborhood \( U_p \in \tau_1 \) and a function \( g \in C \) such that \( f \big| U_p = g \big| U_p \). Since \( \tau_1 \subseteq \tau_2 \), then \( U_p \in \tau_2 \) and \( f \in C_{(M, \tau_2)} \). Consequently \( C_{(M, \tau_1)} \subseteq C_{(M, \tau_2)} \). ■

**Lemma 3.4.** Let \( C = \{f_1, f_2, f_3, \ldots, f_n, n \in \mathbb{N}\} \) and \( D = \{g_1, g_2, g_3, \ldots, g_n, n \in \mathbb{N}\} \) be two sets of real-valued functions on a nonempty set \( M \) and let \( \tau \) be a topology on \( M \). If \( C \subseteq D \), then \( C_M \subseteq D_M \).

**Proof.** Let \( f \in C_M \). Then for each \( p \in M \), there exist a neighborhood \( U_p \in \tau \) and a function \( g \in C \) such that \( f \big| U_p = g \big| U_p \). Since \( C \subseteq D \), then \( g \in D \). Hence \( f \in D_M \). Thus \( C_M \subseteq D_M \). ■
Lemma 3.5. Let \( C = \{f_1, f_2, f_3, \ldots, f_n, n \in N\} \) and \( D = \{g_1, g_2, g_3, \ldots, g_n, n \in N\} \) be two sets of real–valued functions on a set \( M \) and let \( \tau_1, \tau_2 \) be two topologies on \( M \). If \( C \subseteq D \) and \( \tau_1 \subseteq \tau_2 \), then \( C_{(M, \tau_1)} \subseteq D_{(M, \tau_2)} \).

Proof. From Lemma 3.4, we have \( C_{(M, \tau_1)} \subseteq D_{(M, \tau_1)} \). (1)

By Lemma 3.3, we have \( D_{(M, \tau_1)} \subseteq D_{(M, \tau_2)} \). (2)

From (1) and (2), we have \( C_{(M, \tau_1)} \subseteq D_{(M, \tau_2)} \). ■

4. Composition with Euclidean Smooth Functions.

Let \( C = \{f_1, f_2, f_3, \ldots, f_n, n \in N\} \) be a set of real–valued functions defined on a nonempty set \( M \). As in [8, 11 and 14], the set \( scC \) is defined by setting \( f \in scC \) if and only if there exist \( f_1^*, f_2^*, \ldots, f_n^* \in C, n \in N \), and a function \( \omega: \mathbb{R}^n \rightarrow \mathbb{R} \) of class \( C^\infty \) such that

\[
f = \omega \circ (f_1^*, f_2^*, \ldots, f_n^*).
\]

In other words [4, 5, 6, 10, 11 and 14]:

\[
scC = \{ \omega \circ (f_1^*, f_2^*, \ldots, f_n^*): f_1^*, f_2^*, \ldots, f_n^* \in C, \omega \in C^\infty (\mathbb{R}^n, \mathbb{R}), n \in N \}.
\]

Lemma 4.1. Let \( C = \{f_1, f_2, f_3, \ldots, f_n, n \in N\} \) be a set of real–valued functions defined on a nonempty set \( M \). Then \( C \subseteq scC \).

Proof. Let \( f \in C \) and let \( I_\mathbb{R}: \mathbb{R} \rightarrow \mathbb{R} \) be the identity function. Since \( I_\mathbb{R} \in C^\infty (\mathbb{R}, \mathbb{R}) \), then \( I_\mathbb{R} \circ f \in scC \). Since \( f = I_\mathbb{R} \circ f \), then \( f \in scC \). Hence \( C \subseteq scC \). ■

Corollary 4.1. Let \( C = \{f_1, f_2, f_3, \ldots, f_n, n \in N\} \) be a set of real–valued functions on a nonempty set \( M \). Then \( scC = C \) if and only if \( scC \subseteq C \).

Proof. If \( scC = C \), then \( scC \subseteq C \). Conversely, if \( scC \subseteq C \), then by the preceding lemma we have \( C \subseteq scC \). Consequently \( scC = C \). ■

The following definition is given in [4, 11, 12 and 14].

Definition 4.1. Let \( C = \{f_1, f_2, f_3, \ldots, f_n, n \in N\} \) be a set of real–valued functions over a nonempty set \( M \). If \( scC = C \), then \( C \) is said to be closed with respect to composition with Euclidean smooth functions.

Lemma 4.2. Let \( C_1, C_2 \) be two sets of real–valued functions on a nonempty set \( M \). If \( C_1 \subseteq C_2 \), then \( scC_1 \subseteq scC_2 \).
Proof. Let \( f \in scC_1 \). Then \( f = \omega \circ (f_1, f_2, \ldots, f_n) \) for some \( f_1, f_2, \ldots, f_n \in C_1, \ n \in \mathbb{N} \), and some \( \omega \in C^n (\mathbb{R}^n, \mathbb{R}) \). Since \( C_1 \subseteq C_2 \), then \( f_1, f_2, \ldots, f_n \in C_2 \) and \( f \in scC_2 \). Hence \( scC_1 \subseteq scC_2 \). ■

**Example 4.1.** Let \( C = \{f_1, f_2, f_3, \ldots, f_n : n \in \mathbb{N}\} \) be the set of all continuous real-valued functions on a topological space \( M \). Then \( scC \subseteq C \).

**proof.** Let \( f \in scC \). Then there exist \( f_1, f_2, \ldots, f_n \in C, \ n \in \mathbb{N} \), and a function \( \omega \in C^n (\mathbb{R}^n, \mathbb{R}) \) such that \( f = \omega \circ (f_1, f_2, \ldots, f_n) \). Then \( f \) is continuous because \( \omega \) is continuous, \( (f_1, f_2, \ldots, f_n) \) is a continuous function and the composition of two continuous functions is a continuous function. This means that \( f \in C \). Hence \( scC \subseteq C \). ■

**Theorem 4.1.** Let \( C_0 = \{f_1, f_2, f_3, \ldots, f_n : n \in \mathbb{N}\} \) be a set of real-valued functions defined on a nonempty set \( M \). Then the initial topology with respect to \( C_0 \) and \( scC_0 \) coincide, i.e., \( \tau_{C_0} = \tau_{scC_0} \).

**proof.** Let \( f \in C_0 \). By Lemma 4.1, \( C_0 \subseteq scC_0 \). Then \( f \in scC_0 \) and \( f: (M, \tau_{scC_0}) \to \mathbb{R} \) is continuous for each \( f \in C_0 \). But \( \tau_{C_0} \) is the weakest topology such that each \( f \in C_0 \) is continuous. Hence \( \tau_{C_0} \subseteq \tau_{scC_0} \). (3)

On the other hand, let \( g \in scC_0 \). Then \( g = \omega \circ (f_1^*, f_2^*, \ldots, f_n^*) \) for some \( f_1^*, f_2^*, \ldots, f_n^* \in C_0 \), where \( n \in \mathbb{N} \), and some \( \omega \in C^n (\mathbb{R}^n, \mathbb{R}) \). Since the topology of \( \mathbb{R}^n \) is the initial with respect to the projections \( \pi_i : \mathbb{R}^n \to \mathbb{R} \), \( \pi_i (x_1, x_2, \ldots, x_n) = x_i \), and \( \pi_i \circ (f_1^*, f_2^*, \ldots, f_n^*) = f_i : (M, \tau_{C_0}) \to \mathbb{R} \) is continuous, then \( (f_1^*, f_2^*, \ldots, f_n^*): (M, \tau_{C_0}) \to \mathbb{R}^n \) is continuous. So the composition \( g = \omega \circ (f_1^*, f_2^*, \ldots, f_n^*): (M, \tau_{C_0}) \to \mathbb{R} \) of two continuous functions is continuous. Hence \( g: (M, \tau_{C_0}) \to \mathbb{R} \) is continuous for all \( g \in scC_0 \). But \( \tau_{scC_0} \) is the weakest topology on \( M \) such that each \( g \in scC_0 \) is continuous. It follows that \( \tau_{scC_0} \subseteq \tau_{C_0} \). (4)

From (3) and (4), we have \( \tau_{C_0} = \tau_{scC_0} \). ■

In general, by Lemma 3.1, \( C \subseteq C_M \) nevertheless the following theorem says that \( \tau_C = \tau_{C_M} \).
Theorem 4.2. Let \( C = \{f_1, f_2, f_3, \ldots, f_n, n \in N\} \) be a set of real-valued functions defined on a nonempty set \( M \). Then \( \tau_C = \tau_{\text{C}_M} \), where \( \text{C}_M \) is the set of all local \( C \)-functions with respect to the topology \( \tau_C \) on \( M \).

**Proof.** Let \( f \in \text{C}_M \). Then for each \( p \in M \), there exist a neighborhood \( U_p \in \tau_C \) of \( p \) and a function \( g_p \in C \) such that \( f|_U = g_p|_U \). Since \( g_p : (M, \tau_C) \to \mathbb{R} \) is continuous, then \( f : (M, \tau_C) \to \mathbb{R} \) is continuous at \( p \). Since \( p \in M \) is arbitrary, then \( f : (M, \tau_C) \to \mathbb{R} \) is continuous. But this means that \( f : (M, \tau_C) \to \mathbb{R} \) is continuous for any \( f \in \text{C}_M \). Since \( \tau_{\text{C}_M} \) is the weakest topology on \( M \) such that each \( f \in \text{C}_M \) is continuous, then \( \tau_{\text{C}_M} \subseteq \tau_C \). \( \Box \)

Conversely, suppose that \( f \in C \). Since, by Lemma 3.1, we have \( C \subseteq \text{C}_M \), then \( f \in \text{C}_M \). Therefore \( f : (M, \tau_{\text{C}_M}) \to \mathbb{R} \) is continuous for all \( f \in C \). But \( \tau_C \) is the weakest topology on \( M \) such that \( f \) is continuous for all \( f \in C \). Consequently \( \tau_C \subseteq \tau_{\text{C}_M} \). \( \Box \)

From (5) and (6), we have \( \tau_C = \tau_{\text{C}_M} \). \( \Box \)

Theorem 4.3. Let \( C = \{f_1, f_2, f_3, \ldots, f_n, n \in N\} \) be a set of real-valued functions defined on a nonempty set \( M \) and consider \( M \) with the topology \( \tau_C \). Then \( \tau_C = \tau_{\text{scC}} = \tau_{\text{C}_M} = \tau_{(\text{scC})_M} = \tau_{\text{scC}_M} \).

**Proof.** From Theorem 4.1 we have \( \tau_C = \tau_{\text{scC}} \) \( \Box \)

And \( \tau_{\text{C}_M} = \tau_{(\text{scC})_M} \). \( \Box \)

And by Theorem 4.2, we have \( \tau_C = \tau_{\text{C}_M} \) \( \Box \)

And \( \tau_{\text{scC}} = \tau_{(\text{scC})_M} \). \( \Box \)

From (7), (8), (9) and (10), we have \( \tau_C = \tau_{\text{scC}} = \tau_{\text{C}_M} = \tau_{(\text{scC})_M} = \tau_{\text{scC}_M} \). \( \Box \)

Theorem 4.4. Let \( C_0 = \{f_1, f_2, f_3, \ldots, f_n, n \in N\} \) be a set of real-valued functions defined on a nonempty set \( M \). If \( C = \langle \text{scC}_0 \rangle_{(M, \tau_{C_0})} \) with respect to the topology \( \tau_{C_0} \), then \( C_{(M, \tau_C)} \subseteq C \) and \( C \) is closed with respect to Localization.
Proof. Let \( f \in C_{(M, \tau_c)} \). That is, for each \( p \in M \), there exist a neighborhood \( U_p \in \tau_c \) of \( p \) with \( p \in U_p \in \tau_c \) and a function \( g_p \in C \) such that \( f|_{U_p} = g_p|_{U_p} \). (11)

Since \( g_p \in C = C_{(M, \tau_{C_0})} \), there exist a neighborhood \( V_p \in \tau_{C_0} \) of \( p \) and a function \( h_p \in scC_0 \) with \( g_p|_{V_p} = h_p|_{V_p} \). (12)

Since \( h_p \in scC_0 \), then \( h_p = \omega \circ (f_1^*, f_2^*, ..., f_n^*) \) (13) for some finite sequence \( f_1^*, f_2^*, ..., f_n^* \in C_0 \), and some \( \omega \in C^\omega (\mathbb{R}^n, \mathbb{R}) \). From (11), (12) and (13), we have \( f|_{U_p \cap V_p} = \omega \circ (f_1^*, f_2^*, ..., f_n^*)|_{U_p \cap V_p} \).

Since \( \tau_{C_0} = \tau_c \), then \( U_p \cap V_p \in \tau_{C_0} \) is an open neighborhood of \( p \). This implies that \( f \in (scC_0)_{(M, \tau_{C_0})} = C \). Hence \( C_{(M, \tau_c)} \subseteq C \). By Lemma 3.2, \( C \) is closed with respect to Localization.

Theorem 4.5. Let \( C^0 = \{f_1, f_2, f_3, ..., f_n, n \in \mathbb{N}\} \) be a set of real-valued functions on a nonempty set \( M \) and consider \( M \) with the topology \( \tau_{C_0} \). Then the set of all local \( scC_0 \)-functions on \( M \), \( C = (scC_0)_M \), is closed with respect to composition with smooth functions, i.e., \( scC = C \).

Proof. Let us observe that \( \tau_{C_0} = \tau_{scC_0} = \tau_c \). Let \( f \in scC \).

We want to show that \( f \in C \). Since \( f \in scC \), then there exist \( f_1^*, f_2^*, ..., f_n^* \in C \) and \( \omega \in C^\omega (\mathbb{R}^n, \mathbb{R}) \) such that \( f = \omega \circ (f_1^*, f_2^*, ..., f_n^*) \). (14)

Since \( f_1^*, f_2^*, ..., f_n^* \in C = (scC_0)_M \), then for each \( p \in M \) there exist neighborhoods \( U_1, U_2, ..., U_n \) of \( p \) and functions \( g_1, g_2, ..., g_n \in scC_0 \) such that \( f_1|_{U_1} = g_1|_{U_1}, ..., f_n|_{U_n} = g_n|_{U_n} \). Let \( U_p = U_1 \cap U_2 \cap ... \cap U_n \). Then

\[
f_1|_{U_p} = g_1|_{U_p}, ..., f_n|_{U_p} = g_n|_{U_p}.
\] (15)

Since \( g_1, g_2, ..., g_n \in scC_0 \), then

\[
g_1 = \omega_1 \circ (g_1^1, ..., g_1^k) \]
\[
\vdots
\]
\[
g_n = \omega_n \circ (g_n^1, ..., g_n^k)
\] (16)
for some $\omega_i \in C^\infty(\mathbb{R}^{k_i}, \mathbb{R}), \ldots, \omega_n \in C^\infty(\mathbb{R}^{k_n}, \mathbb{R})$ and $g_1, \ldots, g_{k_1}, \ldots, g_n, \ldots, g_{k_n} \in C_0$. From (14), (15) and (16), it follows that
\[ f|_{U_p} = \omega \circ \left( \omega_1 \circ \left( g_1, \ldots, g_{k_1} \right), \ldots, \omega_n \circ \left( g_n, \ldots, g_{k_n} \right) \right)|_{U_p}. \]
Thus
\[ f|_{U_p} = \omega \circ \left( \theta_1, \theta_2, \ldots, \theta_n \right) \circ \left( g_1, \ldots, g_{k_1}, \ldots, g_n, \ldots, g_{k_n} \right)|_{U_p} \]
where $\theta_1, \theta_2, \ldots, \theta_n$ are functions in $C^\infty(\mathbb{R}^{k_1+\ldots+k_n}, \mathbb{R})$ defined by
\[
\begin{align*}
\theta_1(x_1, x_1^1, x_n, x_n^1, \ldots, x_n^{k_n}) &= \omega_1(x_1, x_1^1), \\
&\vdots \\
\theta_n(x_1, x_1^k, x_1^1, \ldots, x_n^1, \ldots, x_n^{k_n}) &= \omega_n(x_n, x_n^k).
\end{align*}
\]
Let $g_p = \omega \circ \left( \theta_1, \ldots, \theta_n \right) \circ \left( g_1, \ldots, g_{k_1}, \ldots, g_n, \ldots, g_{k_n} \right)$. Then, for each $p \in M$, there exist a neighborhood $U_p$ of $p$ and a function $g_p \in scC_0$ such that $f|_{U_p} = g_p|_{U_p}$. Consequently
\[ f \in (scC_0)_M = C. \]
This is true for all $f \in scC$. Hence $scC \subseteq C$. This completes the proof.

References


