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Properties of Certain Subclasses of P−valent Functions Defined by Certain Integral Operator

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الملخص:

في هذا البحث، قمنا بتقديم تعريفات قياسية لفصول جزئية جديدة من الدوال متعددة التكافؤ المنتظمة النجمية والمنتظمة المحدبة، ودراسة بعض الخصائص لفصول معينة من الدوال التحليلية المنتظمة المعرفة بواسطة مؤثر تكاملي معين.

الكلمات الدالة: دوال تحليلية؛ دوال متعددة التكافؤ؛ دوال نجمية؛ دوال محدبة؛ دوال نجمية منتظمة؛ دوال محدبة منتظمة.

Abstract

In this paper, we introduce standard definitions for new subclasses of uniformly starlike, uniformly convex p−valent functions, and study some properties of uniformly certain classes of analytic functions defined by certain integral operator.

Keywords: Analytic functions; p−valend functions; starlike functions; convex functions; uniformly starlike functions; uniformly convex functions.

Introduction1−

Let $H(U)$ be the class of analytic functions in the open unit disk $U=\{z: z\in\mathbb{C}, |z|<1\}$ and $H[a,p]$ be subclass of $H(U)$ consisting of functions of the form:

$$f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\cdots \quad (a\in\mathbb{C}; n\in\mathbb{N}=\{1,2,\ldots\}),$$

Also, let $A(p)$ be the subclass of the functions $f\in H(U)$ of the form:

$$f(z)=z^p+\sum_{n=p+1}^{\infty} a_nz^n \quad (p\in\mathbb{N}), \qquad (1)$$

we note that $A(1)=A$ the class of univalent functions.

Let $f\in A(p)$ given by (1) and $g\in A(p)$, given by

$$g(z)=z^p+\sum_{n=p+1}^{\infty} b_nz^n .$$

The Hadamard product (or convolution) of f and g is given by

$$(f*g)(z)=z^p+\sum_{n=2}^{\infty} a_n b_n z^n=(g*f)(z).$$

For the functions, $f,g\in H(U)$, we say that f is *subordinate* to g, written $f\prec g$ or $f(z)\prec g(z)$, if there exists a Schwarz function $w(z)$, which is analytic in U with $w(0)=0$ and $|w(z)|<1$ $(z\in U)$ such that $f(z)=g(w(z))$ $(z\in U)$ (see[12], [16]).

Furhermore, if g is univalent in U, then we have the following equivalence:

$f(z)\prec g(z)\Leftrightarrow f(0)=g(0)$ and $f(U)\subset g(U)$.

Definition 1. [1, 9, 14].

A function $f\in A(p)$ is called *p−valent starlike of order α* if $f(z)$ satisfies the following condition:

$$Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \quad (0\le\alpha<p;\ p\in\mathbb{N};\ z\in U), \qquad (2)$$

we denote by $S_p^*(\alpha)$ the class of all p−valent starlike functions of order α.

Definition 2 [14, 18, 19].

A function $f\in A(p)$ is called *p−valent convex of order α* if $f(z)$ satisfies the following condition:

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha \quad (0\le\alpha<p;\ p\in\mathbb{N};\ z\in U), \qquad (3)$$

we denote by $K_p(\alpha)$ the class of all p−valent convex functions of order α .

The classes $S_p^*(\alpha)$ and $K_p(\alpha)$ were studied by Patil and Thakare[19] and Owa[18].

Further from (2) and (3), we can see that:

$$f(z)\in K_p(\alpha)\Leftrightarrow \frac{zf'(z)}{p}\in S_p^*(\alpha). \qquad (4)$$

We denote by $S_p^*=S_p^*(0)$ and $K_p=K_p(0)$, where S_p^* and K_p are the class of p−valent starlike functions and p−valent convex functions, respectively, (see[15]).

Definition 3 [18, 19].

A function $f\in A(p)$ is called *p−valent close−to−convex of order α* if $f(z)$ satisfies the following condition:

$$Re\left\{\frac{zf'(z)}{g(z)}\right\}>\alpha \quad \left(0\le\alpha<p;\ g\in S_p^*;\ z\in U\right), \qquad (5)$$

we denote by $C_p(\alpha)$ the class of all p−valent close−to−convex functions of order α.

Definition 4 [1, 18].

A function $f\in A(p)$ is called *p−valent quasi−convex of order α* if $f(z)$ satisfies the following condition:

$$Re\left\{\frac{(zf'(z))'}{g'(z)}\right\}>\alpha \quad \left(0\le\alpha<p;\ g\in K_p;\ z\in U\right), \qquad (6)$$

we denote by $C_p^*(\alpha)$ the class of all p−valent quasi−convex functions of order α.

Definition 5 [1, 4].

A function $f\in A(p)$ is said to be in $SP_p(\alpha)$, the class of *uniformly p−valent starlike functions of order α* if it satisfies the condition:

$$Re\left\{\frac{zf'(z)}{f(z)}\right\}\ge\left|\frac{zf'(z)}{f(z)}-p\right|+\alpha. \qquad (7)$$

Definition 6 [1, 4].

A function $f \in A(p)$ is said to be in $UCV_p(\alpha)$, the class of *uniformly p−valent convex functions of order* α if it satisfies the condition:

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geq \left|1+\frac{zf''(z)}{f'(z)}-p\right| + \alpha. \qquad (8)$$

Definition 7 [1, 3, 4].

A function $f(z)$ of the form (1) is said to be in the class of *β−uniformly p−valent starlike functions*, denoted by $\beta - UST_p$, if it satisfies the following condition:

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta\left|\frac{zf'(z)}{f(z)}-p\right| \quad (\beta \geq 0; z \in U). \qquad (9)$$

Definition 8 [1, 3, 4].

A function $f(z)$ of the form (1) is said to be in the class of *β−uniformly p−valent convex functions*, denoted by $\beta - UCV_p$, if it satisfies the following condition:

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \beta\left|1+\frac{zf''(z)}{f'(z)}-p\right| \quad (\beta \geq 0;\ z \in U). \qquad (10)$$

Definition 9 [3, 4].

A function $f \in A(p)$ is said to be in $UKC_p(\alpha)$, the class of *uniformly p−valent close−to−convex functions of order* α if it satisfies the condition:

$$Re\left\{\frac{zf'(z)}{g(z)}\right\} \geq \left|\frac{zf'(z)}{g(z)}-p\right| + \alpha, \qquad (11)$$

for some $g(z) \in SP_p(\alpha)$.

Definition 10 [3, 4].

A function $f \in A(p)$ is said to be in $UQC_p(\alpha)$, the class of *uniformly p−valent quasi−convex functions of order* α if it satisfies the condition:

$$Re\left\{\frac{zf'(z)}{g'(z)}\right\} \geq \left|\frac{zf'(z)}{g'(z)}-p\right| + \alpha, \qquad (12)$$

Definition 11 [3, 4, 13].

A function $f \in A(p)$ is said to be in $SP_p(\beta,\alpha)$, the class of *uniformly p−valent starlike functions of order* α *and type* β if it satisfies the condition:

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} \geq \beta\left|\frac{zf'(z)}{f(z)}-p\right| + \alpha \quad (\beta \geq 0, 0 \leq \alpha < p). \qquad (13)$$

Definition 12 [3, 4, 13].

A function $f \in A(p)$ is said to be in $UCV_p(\beta,\alpha)$, the class of *uniformly p−valent convex functions of order* α *and type* β if it satisfies the condition:

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geq \beta\left|1+\frac{zf''(z)}{f'(z)}-p\right|+\alpha \qquad (\beta \geq 0, 0 \leq \alpha < p). \quad (14)$$

Lemma 1 [1, 6, 10].

Let a, b be complex constants and $h(z)$ be univalent convex function in U with $h(0)=p$ and $Re(ah(z)+b)>0$.

Let $g(z)$ be analytic in U. Then

$$g(z)+\frac{zg'(z)}{ag(z)+b} \prec h(z) \text{ in } U,$$

implies $p(z) \prec h(z)$ in U.

Lemma 2 [6, 7].

Let $h(z)$ be convex univalent in U and $Q(z)$ be analytic in U with $Re\{Q(z)\}>0,\ (z \in U)$.

If $p(z)$ is analytic in U with $p(0)=h(0)=p$, then

$p(z)+Q(z)zp'(z) \prec h(z),\ (z \in U)$

implies $p(z) \prec h(z)$ in $U,\ (z \in U)$.

2. Geometric Interpretation

Let $f, SP_P(\beta,\alpha)$ and $UCV_P(\beta,\alpha)$ are given by (1), (13) and (14), respectively, then $f \in SP_P(\beta,\alpha)$ and $f \in UCV_P(\beta,\alpha)$ if and only if $\frac{zf'(z)}{f(z)}$ and $1+\frac{zf''(z)}{f'(z)}$, respectively, takes all the values in conic domain $R_{\beta,\alpha}$ which is included in the right half plane (see[2],[4]) given by

$R_{\beta,\alpha}=\{w=u+iv \in \mathbb{C}: u>\beta\sqrt{(u-p)^2+v^2}+\alpha,\ \beta \geq 0$ and $\alpha \in [0,1)\}$. (15)

We note that

(i) For $\beta>1$, if $f \in SP_P(\beta,\alpha)$, then $zf'(z)/f(z)$ lies in region $R_{\beta,\alpha}$ such that

$$R_{\beta,\alpha}=\{w=u+iv \in \mathbb{C}: u>\beta\sqrt{(u-p)^2+v^2}+\alpha,\ \beta \geq 0 \text{ and } \alpha \in [0,1]\},$$

that is, part of the complex plane which contains $w=1$ and is bounded by the ellipse

$$\left(u-(\beta^2p-\alpha)/(\beta^2-1)\right)^2+(\beta^2/(\beta^2-1))v^2=\beta^2(p-\alpha)^2/(\beta^2-1)^2,$$

with the vertices at the points

$A((\beta p+\alpha)/(\beta+1),0); B((\beta p-\alpha)/(\beta-1),0);$

$C\left((\beta^2p-\alpha)/(\beta^2-1),(p-\alpha)/\sqrt{\beta^2-1}\right);$

$D\left((\beta^2p-\alpha)/(\beta^2-1),(\alpha-p)/\sqrt{\beta^2-1}\right).$

Since

$$\alpha<(\beta p+\alpha)/\beta+1)<1<(\beta p-\alpha)/(\beta-1),$$

we have

$R_{\beta,\alpha} \subset \{w: Re\, w>\alpha\}$ and so $R_{\beta,\alpha} \subset S^*(\alpha)$,

Figure 1 shows this region.

For $\beta > 1$

$$(u - \frac{\beta^2 p - \alpha}{\beta^2 - 1})^2 + \frac{\beta^2 v^2}{\beta^2 - 1} = \frac{\beta^2 (p-\alpha)^2}{(\beta^2 - 1)^2}$$

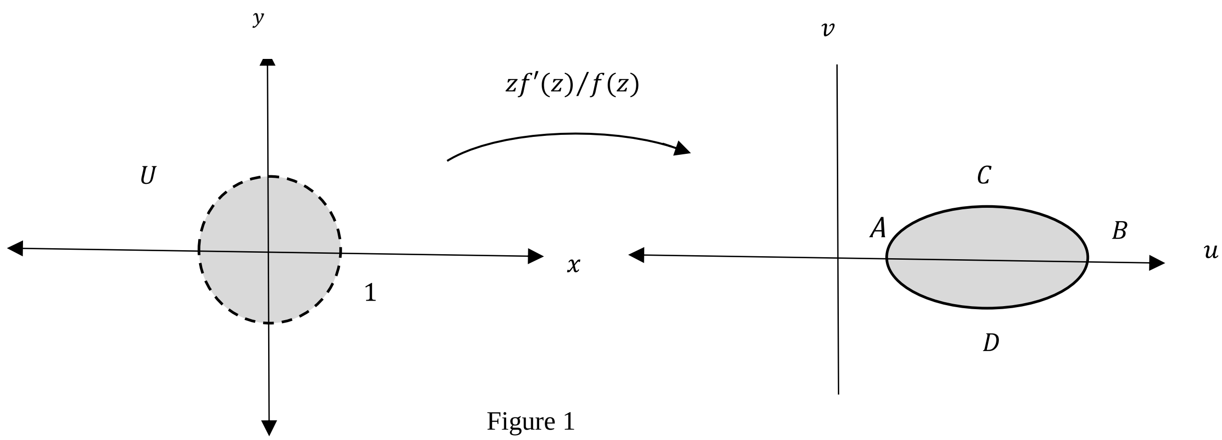

Figure 1

ii) For $\beta = 1$, $f \in SP_P(\beta, \alpha)$, then $zf'(z)/f(z)$ belongs to the region $R_{\beta,\alpha}$ which contains $w = 2$ and is bounded by the parabola
$u = (v^2 + p^2 - \alpha^2)/2(p - \alpha)$ with the vertic at the point $A((p+\alpha)/2, 0)$, as shown in figure 2.

For $\beta = 1$

$$\frac{v^2}{2(p-\alpha)} < (u - \frac{p+\alpha}{2})$$

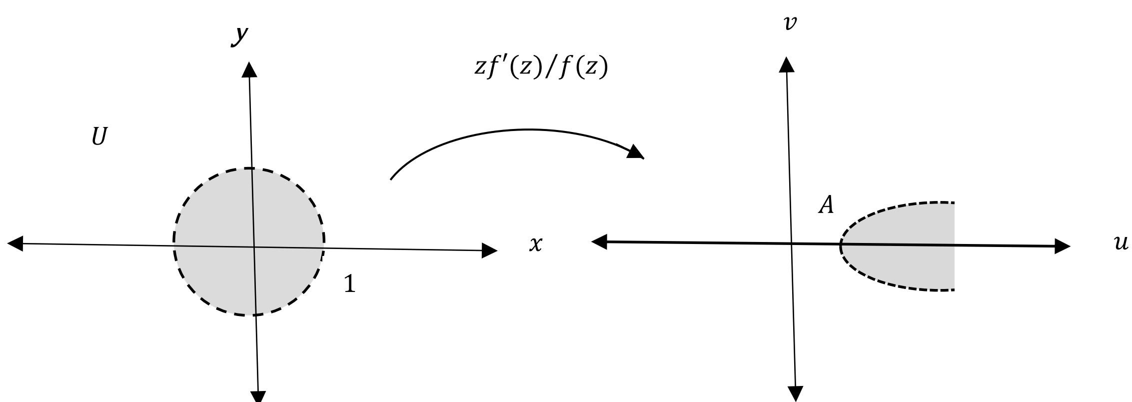

Figure 2

iii) For $\beta > 1$, if $f \in UCV_P(\beta,\alpha)$, then $zf''(z)/f'(z)$ lies in the region $R_{\beta,\alpha}$ such that $R_{\beta,\alpha} = \{w = u + iv \in \mathbb{C}: u > \beta\sqrt{(u-(p-1))^2 + v^2} + \alpha - 1,\ \beta \geq 0$ and $\alpha \in [0,1]\}$, that is, part of the complex plane which contains $w = 1$ and is bounded by the ellipse

$$(u - ((1-\alpha) + \beta^2(p-1))/(\beta^2-1))^2 + (\beta^2/(\beta^2-1))v^2 = \frac{\beta^2(\alpha^2+p^2-2\alpha)}{(\beta^2-1)^2}$$

with the vertices at the points

$$A\left((-\beta\sqrt{\alpha^2+p^2-2\alpha} + (1-\alpha) + \beta^2(p-1))/(\beta^2-1)\,, 0\right);$$
$$B\left((\beta\sqrt{\alpha^2+p^2-2\alpha} + (1-\alpha) + \beta^2(p-1))/(\beta^2-1)\,, 0\right);$$
$$C\left(((1-\alpha) + \beta^2(p-1))/(\beta^2-1)\,, \sqrt{(\alpha^2+p^2-2\alpha)/(\beta^2-1)}\right);$$
$$D\left(((1-\alpha) + \beta^2(p-1))/(\beta^2-1)\,, -\sqrt{(\alpha^2+p^2-2\alpha)/(\beta^2-1)}\right),$$

Figure 3 show this region.

For $\beta > 1$

$$(u - \frac{(1-\alpha)+\beta^2(p-1)}{\beta^2-1})^2 + \frac{\beta^2 v^2}{\beta^2-1} = \frac{\beta^2(\alpha^2+p^2-2\alpha)}{(\beta^2-1)^2}$$

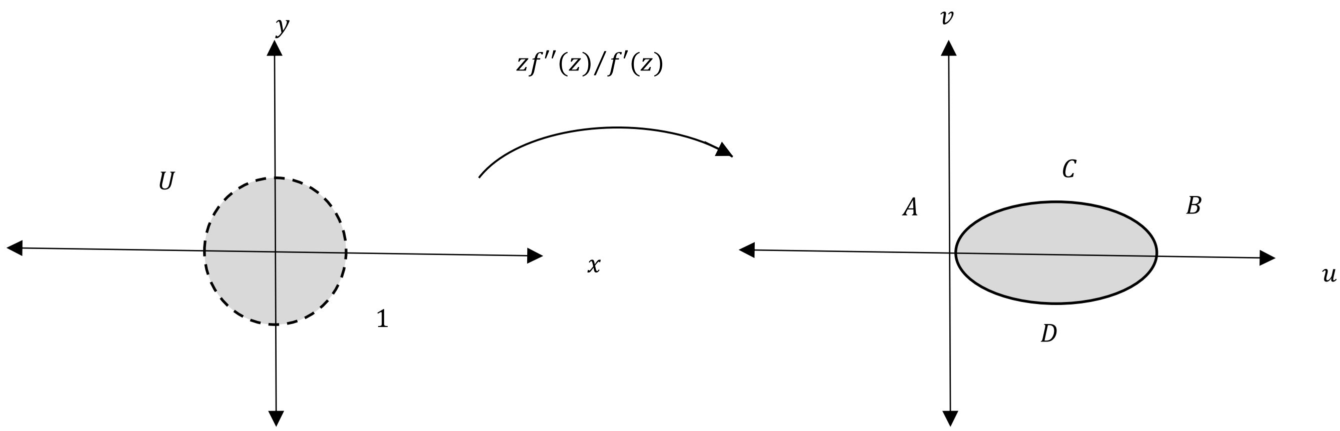

Figure 3

(iv) For $\beta = 1$, if $UCV_P(\beta,\alpha)$, then $zf''(z)/f'(z)$ belongs to the region $R_{\beta,\alpha}$ which contains $w = 2$ and is bounded by the $u = (v^2 + 2(\alpha - p) + p^2 - \alpha^2)/2(p-\alpha)$,

with the vertic at the point $A((p+\alpha-2)/2\,, 0)$, s shown in figure 4.

For $\beta = 1$

$$\frac{v^2}{2(p-\alpha)} < (u + \frac{2-\alpha-p}{2})$$

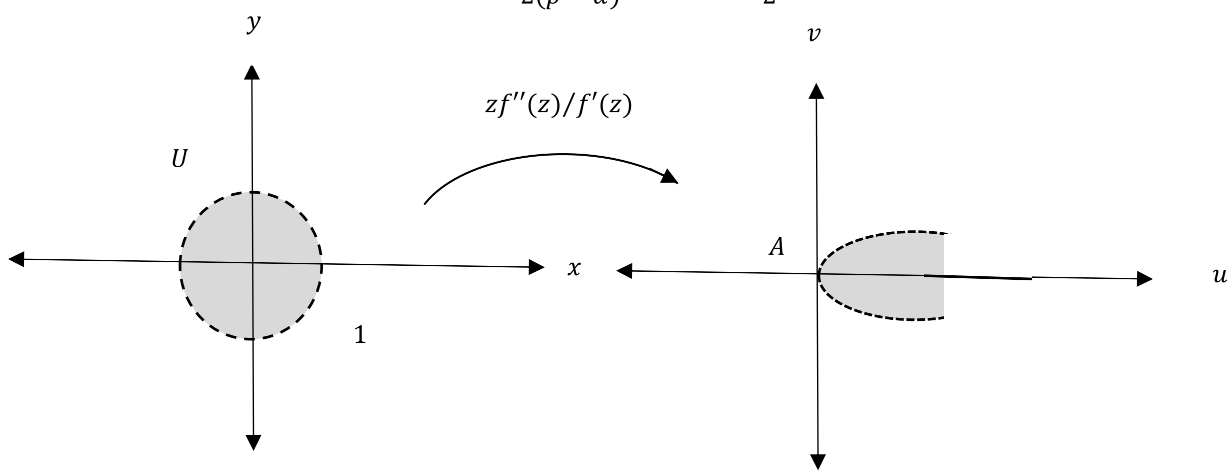

Figure 4

Recently, Komatu ([5], [17]) introduced a certain integral operator $L_a^\lambda (a > 0;\ \lambda \geq 0)$ defined by

$L_a^\lambda f(z) = \frac{a^\lambda}{\Gamma(\lambda)} \int_0^1 t^{a-2} (log \frac{1}{t})^{\lambda-1} f(zt)dt \ \ (z \in U; a > 0; \lambda \geq 0; f(z) \in A) \quad (16)$

Thus, if $f(z) \in A(p)$ is of the from (1), it is easily from (16) that (see [6], [11], [14])

$$L_a^\lambda f(z) = z^p + \sum_{n=p+1}^{\infty} \left(\frac{a-1+p}{a+n-1}\right)^\lambda a_n z^n \quad (a > 0;\ \lambda \geq 0) \qquad (17)$$

Using the above relation, it is easy verify that

$$z(L_a^{\lambda+1} f(z))' = (a-1+p) L_a^\lambda f(z) - (a-1) L_a^{\lambda+1} f(z) \quad (a > 0; \lambda \geq 0) \qquad (18)$$

3− Inclusion Relationships for New Subclasses of P−valent Functions

Let f be given by (1) and the integral operator $L_a^\lambda : A(p) \to A(p)$ be given by (17), (see[8],[13])

Now, we define new classes by using the operator L_a^λ as follows:

Definition 13.

Let $f(z) \in A(p)$ given by (1).Then $f(z) \in SP_{p,a}^\lambda(\beta, \alpha)$ if and only if

$$L_a^\lambda f(z) \in SP_p(\beta, \alpha) \qquad (19)$$

Definition 14.

Let $f(z) \in A(p)$ given by (1).Then $f(z) \in UCV_{p,a}^{\lambda}(\beta,\alpha)$ if and only if

$$L_a^{\lambda} f(z) \in UCV_p(\beta,\alpha) \tag{20}$$

Definition 15.

Let $f(z) \in A(p)$ given by (1). Then $f(z) \in UKC_{p,a}^{\lambda}(\beta,\alpha)$ if and only if

$$L_a^{\lambda} f(z) \in UKC_p(\beta,\alpha) \tag{21}$$

Definition 16.

Let $f(z) \in A(p)$ given by (1).Then $f(z) \in UQC_{p,a}^{\lambda}(\beta,\alpha)$ if and only if

$$L_a^{\lambda} f(z) \in UQC_p(\beta,\alpha) \tag{22}$$

Theorem 1.

Let $f(z) \in A(p);\ a > 0,\ \beta \geq 0,\ 0 \leq \alpha < p,\ \lambda \geq 0$. Then

$$SP_{p,a}^{\lambda}(\beta,\alpha) \subset SP_{p,a}^{\lambda+1}(\beta,\alpha).$$

Proof.

Let $f(z) \in SP_{p,a}^{\lambda}(\beta,\alpha)$ and suppose that

$$\frac{z\left(L_a^{\lambda+1} f(z)\right)'}{L_a^{\lambda+1} f(z)} = p(z), \tag{23}$$

where $p(z) = p + p_1 z + p_2 z^2 + \cdots$ is analytic in U and $p(z) \neq 0$ for all $z \in U$.

Using the identity (18), we have

$$\frac{(a-1+p) L_a^{\lambda} f(z)}{L_a^{\lambda+1} f(z)} = p(z) + (a-1) \tag{24}$$

By using the logarithmic differentiation on both side of (24) with respect to z, we obtain

$$\frac{z\left(L_a^{\lambda} f(z)\right)'}{L_a^{\lambda} f(z)} = \frac{z\left(L_a^{\lambda+1} f(z)\right)'}{L_a^{\lambda+1} f(z)} + \frac{zp'(z)}{p(z)+(a-1)}$$

which, in view of (23), yields

$$\frac{z\left(L_a^{\lambda} f(z)\right)'}{L_a^{\lambda} f(z)} = p(z) + \frac{zp'(z)}{p(z)+(a-1)} \tag{25}$$

From (25), we see that

$$Re\{h(z) + (a-1)\} > 0, \ \ (z \in U),$$

and

$$p(z) + \frac{zp'(z)}{p(z)+(a-1)} \prec h(z),$$

thus, by using Lemma 1 and (23), we observe that

$$p(z) \prec h(z),$$

so that

$$f(z) \in SP_{p,a}^{\lambda+1}(\beta,\gamma),$$

this implies that

$$SP_{p,a}^{\lambda}(\beta,\alpha) \subset SP_{p,a}^{\lambda+1}(\beta,\alpha).$$

This completes the proof of Theorem 1.

Theorem 2.

Let $f(z) \in A(p); a > 0,\ \beta \geq 0,\ 0 \leq \alpha < p,\ \lambda \geq 0$. Then

$$UCV_{p,a}^{\lambda}(\beta,\alpha) \subset UCV_{p,a}^{\lambda+1}(\beta,\alpha).$$

Proof.

$f(z) \in UCV_{p,a}^{\lambda}(\beta,\alpha) \Leftrightarrow L_a^{\lambda} f(z) \in UCV_p(\beta,\alpha)$

$\Leftrightarrow z\left(L_a^{\lambda} f(z)\right)' \in SP_p(\beta,\alpha)$

$\Leftrightarrow L_a^{\lambda}(zf'(z)) \in SP_p(\beta,\alpha)$

$\Leftrightarrow zf'(z) \in SP_{p,a}^{\lambda}(\beta,\alpha)$

$\Rightarrow zf'(z) \in SP_{p,a}^{\lambda+1}(\beta,\alpha)$

$\Leftrightarrow L_a^{\lambda+1}(zf'(z)) \in SP_p(\beta,\alpha)$

$\Leftrightarrow z\left(L_a^{\lambda+1} f(z)\right)' \in SP_p(\beta,\alpha)$

$\Leftrightarrow L_a^{\lambda+1} f(z) \in UCV_p(\beta,\alpha)$

$\Leftrightarrow f(z) \in UCV_{p,a}^{\lambda+1}(\beta,\alpha)$.

Which evidently proves Theorem 2.

Theorem 3.

Let $f(z) \in A(p); a > 0,\ \beta \geq 0,\ 0 \leq \alpha < p,\ \lambda \geq 0$. Then

$$UKC_{p,a}^{\lambda}(\beta,\alpha) \subset UKC_{p,a}^{\lambda+1}(\beta,\alpha).$$

Proof.

Let $f(z) \in UKC_{p,a}^{\lambda}(\beta,\alpha)$.

Then, by (21), there exists a function $q(z) \in SP_p(\beta,\alpha)\quad (0 \leq \alpha < p)$ such that

$$\frac{z\left(L_a^{\lambda} f(z)\right)'}{q(z)} \prec \psi(z) \quad in\ U \tag{26}$$

Taking

$$q(z) = L_a^{\lambda} g(z) \in SP_p(\beta,\alpha). \tag{27}$$

We find from (19) and (27) that $g(z) \in SP_{p,a}^{\lambda}(\beta,\alpha)$ and

$$\frac{z\left(L_a^{\lambda} f(z)\right)'}{L_a^{\lambda} g(z)} \prec \psi(z) \quad in\ U.$$

Respectively. We now let

$$\frac{z\left(L_a^{\lambda+1} f(z)\right)'}{L_a^{\lambda+1} g(z)} = p(z). \tag{28}$$

Where $\qquad p(z) = p + p_1 z + p_2 z^2 + \cdots$

Making use of the operator identity (18), we also have

$$\frac{z\left(L_a^{\lambda} f(z)\right)'}{L_a^{\lambda} g(z)} = \frac{L_a^{\lambda}(zf'(z))}{L_a^{\lambda} g(z)}$$

$$= \frac{z\left(L_a^{\lambda+1}(zf'(z))\right)' + (a-1)L_a^{\lambda+1}(zf'(z))}{z\left(L_a^{\lambda+1} g(z)\right)' + (a-1)L_a^{\lambda+1} g(z)}$$

By Theorem 1, we know that

$$g(z) \in SP_{p,a}^{\lambda}(\beta,\alpha)$$

and

$$SP_{p,a}^{\lambda}(\beta,\alpha) \subset SP_{p,a}^{\lambda+1}(\beta,\alpha) \tag{29}$$

so that we can put

$$\frac{z\left(L_a^{\lambda+1}g(z)\right)'}{L_a^{\lambda+1}g(z)} = G(z) \tag{30}$$

where

$$Re\{G(z)\} > 0, \quad (z \in U).$$

Upon substituting from (28) and (30) into (29), we have

$$\frac{z\left(L_a^{\lambda}f(z)\right)'}{L_a^{\lambda}g(z)} = \frac{\frac{z\left(L_a^{\lambda+1}(zf`(z))\right)'}{L_a^{\lambda+1}g(z)} + (a-1)\frac{L_a^{\lambda+1}(zf'(z))}{L_a^{\lambda+1}g(z)}}{\frac{z\left(L_a^{\lambda+1}g(z)\right)'}{L_a^{\lambda+1}g(z)} + (a-1)}$$

$$= \frac{\frac{z\left(L_a^{\lambda+1}(zf`(z))\right)'}{L_a^{\lambda+1}g(z)} + (a-1)p(z)}{G(z)+(a-1)} \tag{31}$$

Now from (28), we have

$$z\left(L_a^{\lambda+1}f(z)\right)' = p(z)L_a^{\lambda+1}g(z). \tag{32}$$

Differentiating both sides of (32) with respect to z, we have

$$\left(z\left(L_a^{\lambda+1}f(z)\right)'\right)' = L_a^{\lambda+1}g(z)\left(zp'(z)\right) + p(z)z\left(L_a^{\lambda+1}g(z)\right)'$$

$$\frac{z\left(L_a^{\lambda+1}\left(zf'(z)\right)\right)'}{L_a^{\lambda+1}g(z)} = G(z)p(z) + zp'(z). \tag{33}$$

Making use of (26), (31) and (33), we get

$$\frac{z\left(L_a^{\lambda}f(z)\right)'}{L_a^{\lambda}g(z)} = p(z) + \frac{zp'(z)}{G(z)+(a-1)} \prec \psi(z) \tag{34}$$

since

$$Re\{G(z)+a-1\} > 0, \quad (z \in U).$$

Hence, be taking

$$Q(z) = \frac{1}{G(z)+(a-1)}$$

In (34), and applying Lemma 2, we can show that

$$p(z) \prec \psi(z) \quad in\ U$$

so that

$$f(z) \in UKC_{p,a}^{\lambda+1}(\beta,\alpha)$$

this implies that

$$UKC_{p,a}^{\lambda}(\beta,\alpha) \subset UKC_{p,a}^{\lambda+1}(\beta,\alpha).$$

This completes the proof of Theorem 3.

Theorem 4.

Let $f(z) \in A(p)$; $a > 0$, $\beta \geq 0$, $0 \leq \alpha < p$, $\lambda \geq 0$. Then

$$UQC_{p,a}^{\lambda}(\beta,\alpha) \subset UQC_{p,a}^{\lambda+1}(\beta,\alpha).$$

Proof.

$f(z) \in UQC_{p,a}^{\lambda}(\beta,\alpha) \Leftrightarrow L_a^{\lambda}f(z) \in UQC_p(\beta,\alpha)$

$\Leftrightarrow z\left(L_a^{\lambda}f(z)\right)' \in UKC_p(\beta,\alpha)$

$\Leftrightarrow L_a^{\lambda}\left(zf'(z)\right) \in UKC_p(\beta,\alpha)$

$\Leftrightarrow zf'(z) \in UKC_p(\beta, \alpha)$
$\Rightarrow zf'(z) \in UKC_{p,a}^{\lambda+1}(\beta, \alpha)$
$\Leftrightarrow L_a^{\lambda+1}(zf'(z)) \in UKC_p(\beta, \alpha)$
$\Leftrightarrow z\left(L_a^{\lambda+1}f(z)\right)' \in UKC_p(\beta, \alpha)$
$\Leftrightarrow L_a^{\lambda+1}f(z) \in UQC_p(\beta, \alpha)$
$\Leftrightarrow f(z) \in UQC_{p,a}^{\lambda+1}(\beta, \alpha).$

Which evidently proves Theorem 4.

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