Properties of Certain Subclasses of P–valent Functions Defined by Certain Integral Operator

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Abstract
In this paper, we introduce standard definitions for new subclasses of uniformly starlike, uniformly convex p–valent functions, and study some properties of uniformly certain classes of analytic functions defined by certain integral operator.

Keywords: Analytic functions; p–valent functions; starlike functions; convex functions; uniformly starlike functions; uniformly convex functions.

Introduction
Let $H(U)$ be the class of analytic functions in the open unit disk $U = \{z: z \in \mathbb{C}, |z| < 1\}$ and $H[a,p]$ be subclass of $H(U)$ consisting of functions of the form:
$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (a \in \mathbb{C}; n \in \mathbb{N} = \{1,2, \ldots \}),$$
Also, let $A(p)$ be the subclass of the functions $f \in H(U)$ of the form:
$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N}), \quad (1)$$
we note that $A(1) = A$ the class of univalent functions.

Let $f \in A(p)$ given by (1) and $g \in A(p)$, given by
$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n.$$
\[
(f \ast g)(z) = z^p + \sum_{n=2}^{\infty} a_n b_n z^n = (g \ast f)(z).
\]

For the functions, \(f, g \in H(U)\), we say that \(f\) is subordinate to \(g\), written \(f < g\) or \(f(z) < g(z)\), if there exists a Schwarz function \(w(z)\), which is analytic in \(U\) with \(w(0) = 0\) and \(|w(z)| < 1\) (\(z \in U\)) such that \(f(z) = g(w(z))\) (\(z \in U\)) (see [12], [16]).

Furthermore, if \(g\) is univalent in \(U\), then we have the following equivalence:
\[
f(z) < g(z) \iff f(0) = g(0) \text{ and } f(U) \subset g(U).
\]

**Definition 1.** [1, 9, 14].

A function \(f \in A(p)\) is called \(p\)-valent starlike of order \(\alpha\) if \(f(z)\) satisfies the following condition:
\[
\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (0 \leq \alpha < p; \ p \in \mathbb{N}; \ z \in U),
\]
we denote by \(S_p^*(\alpha)\) the class of all \(p\)-valent starlike functions of order \(\alpha\).

**Definition 2** [14, 18, 19].

A function \(f \in A(p)\) is called \(p\)-valent convex of order \(\alpha\) if \(f(z)\) satisfies the following condition:
\[
\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (0 \leq \alpha < p; \ p \in \mathbb{N}; \ z \in U),
\]
we denote by \(K_p(\alpha)\) the class of all \(p\)-valent convex functions of order \(\alpha\).

The classes \(S_p^*(\alpha)\) and \(K_p(\alpha)\) were studied by Patil and Thakare [19] and Owa [18].

Further from (2) and (3), we can see that:
\[
f(z) \in K_p(\alpha) \iff \frac{zf'(z)}{p} \in S_p^*(\alpha).
\]

We denote by \(S_p^* = S_p^*(0)\) and \(K_p = K_p(0)\), where \(S_p^*\) and \(K_p\) are the class of \(p\)-valent starlike functions and \(p\)-valent convex functions, respectively, (see [15]).

**Definition 3** [18, 19].

A function \(f \in A(p)\) is called \(p\)-valent close-to-convex of order \(\alpha\) if \(f(z)\) satisfies the following condition:
\[
\Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha \quad (0 \leq \alpha < p; \ g \in S_p^*; \ z \in U),
\]
we denote by \(C_p(\alpha)\) the class of all \(p\)-valent close-to-convex functions of order \(\alpha\).

**Definition 4** [1, 18].

A function \(f \in A(p)\) is called \(p\)-valent quasi-convex of order \(\alpha\) if \(f(z)\) satisfies the following condition:
\[
\Re\left(\frac{zf'(z)}{g'(z)}\right) > \alpha \quad (0 \leq \alpha < p; \ g \in K_p; \ z \in U),
\]
we denote by \(C_p(\alpha)\) the class of all \(p\)-valent quasi-convex functions of order \(\alpha\).

**Definition 5** [1, 4].

A function \(f \in A(p)\) is said to be in \(SP_p(\alpha)\), the class of uniformly \(p\)-valent starlike functions of order \(\alpha\) if it satisfies the condition:
\[
\Re\left(\frac{zf'(z)}{f(z)}\right) \geq \left|\frac{zf'(z)}{f(z)} - p\right| + \alpha.
\]
Definition 6 [1, 4].

A function \( f \in A(p) \) is said to be in \( UCV_p(\alpha) \), the class of uniformly \( p \)-valent convex functions of order \( \alpha \) if it satisfies the condition:
\[
Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \geq \left| 1 + \frac{zf''(z)}{f'(z)} - p \right| + \alpha.
\] (8)

Definition 7 [1, 3, 4].

A function \( f(z) \) of the form (1) is said to be in the class of \( \beta \)-uniformly \( p \)-valent starlike functions, denoted by \( \beta - UST_p \), if it satisfies the following condition:
\[
Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \beta \left| \frac{zf'(z)}{f(z)} - p \right| \quad (\beta \geq 0; z \in U) .
\] (9)

Definition 8 [1, 3, 4].

A function \( f(z) \) of the form (1) is said to be in the class of \( \beta \)-uniformly \( p \)-valent convex functions, denoted by \( \beta - UCV_p \), if it satisfies the following condition:
\[
Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \beta \left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \quad (\beta \geq 0; z \in U).
\] (10)

Definition 9 [3, 4].

A function \( f \in A(p) \) is said to be in \( UKC_p(\alpha) \), the class of uniformly \( p \)-valent close-to-convex functions of order \( \alpha \) if it satisfies the condition:
\[
Re \left\{ \frac{zf'(z)}{g(z)} \right\} \geq \left| \frac{zf'(z)}{g(z)} - p \right| + \alpha,
\] (11)
for some \( g(z) \in SP_p(\alpha) \).

Definition 10 [3, 4].

A function \( f \in A(p) \) is said to be in \( UQC_p(\alpha) \), the class of uniformly \( p \)-valent quasi-convex functions of order \( \alpha \) if it satisfies the condition:
\[
Re \left\{ \frac{zf'(z)}{g(z)} \right\} \geq \left| \frac{zf'(z)}{g(z)} - p \right| + \alpha,
\] (12)

Definition 11 [3, 4, 13].

A function \( f \in A(p) \) is said to be in \( SP_p(\beta, \alpha) \), the class of uniformly \( p \)-valent starlike functions of order \( \alpha \) and type \( \beta \) if it satisfies the condition:
\[
Re \left\{ \frac{zf'(z)}{f(z)} \right\} \geq \beta \left| \frac{zf'(z)}{f(z)} - p \right| + \alpha \quad (\beta \geq 0, 0 \leq \alpha < p).
\] (13)
Definition 12 [3, 4, 13].

A function $f \in A(p)$ is said to be in $UCV_p(\beta, \alpha)$, the class of uniformly $p$-valent convex functions of order $\alpha$ and type $\beta$ if it satisfies the condition:

$$\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \geq \beta \left| 1 + \frac{zf''(z)}{f'(z)} - p \right| + \alpha \quad (\beta \geq 0, 0 \leq \alpha < p). \quad (14)$$

Lemma 1 [1, 6, 10].

Let $a, b$ be complex constants and $h(z)$ be univalent convex function in $U$ with $h(0) = p$ and $\text{Re}(ah(z) + b) > 0$.

Let $g(z)$ be analytic in $U$. Then

$$g(z) + \frac{zg'(z)}{ag(z) + b} < h(z) \text{ in } U,$$

implies $p(z) < h(z)$ in $U$.

Lemma 2 [6, 7].

Let $h(z)$ be convex univalent in $U$ and $Q(z)$ be analytic in $U$ with $\text{Re}\{Q(z)\} > 0$, $(z \in U)$.

If $p(z)$ is analytic in $U$ with $p(0) = h(0) = p$, then

$$p(z) + Q(z)p'(z) < h(z), \quad (z \in U)$$

implies $p(z) < h(z)$ in $U, \quad (z \in U)$.

2. Geometric Interpretation

Let $f, S_p(\beta, \alpha)$ and $UCV_p(\beta, \alpha)$ are given by (1), (13) and (14), respectively, then $f \in S_p(\beta, \alpha)$ and $f \in UCV_p(\beta, \alpha)$ if and only if $\frac{zf'(z)}{f(z)}$ and $1 + \frac{zf''(z)}{f'(z)}$, respectively, takes all the values in conic domain $R_{\beta, \alpha}$ which is included in the right half plane (see[2],[4]) given by

$$R_{\beta, \alpha} = \{ w = u + iv \in \mathbb{C}: u > \beta \sqrt{(u - p)^2 + v^2 + \alpha}, \quad \beta \geq 0 \text{ and } \alpha \in [0,1) \}. \quad (15)$$

We note that

(i) For $\beta > 1$, if $f \in S_p(\beta, \alpha)$, then $zf'(z)/f(z)$ lies in region $R_{\beta, \alpha}$ such that

$$R_{\beta, \alpha} = \{ w = u + iv \in \mathbb{C}: u > \beta \sqrt{(u - p)^2 + v^2 + \alpha}, \quad \beta \geq 0 \text{ and } \alpha \in [0,1] \},$$

that is, part of the complex plane which contains $w = 1$ and is bounded by the ellipse

$$(u - (\beta^2p - \alpha)/\beta^2 - 1)^2 + (\beta^2/(\beta^2 - 1)v^2 = \beta^2(p - \alpha)^2/(\beta^2 - 1)^2,$$

with the vertices at the points

$A(\beta(p + \alpha)/\beta + 1, 0); B((\beta p - \alpha)/(\beta - 1), 0); C(\beta^2p - \alpha)/(\beta^2 - 1), (p - \alpha)/\beta^2 - 1); D((\beta^2p - \alpha)/(\beta^2 - 1), (\alpha - p)/\sqrt{\beta^2 - 1}).$

Since

$$\alpha < (\beta p + \alpha)/\beta + 1 < 1 < (\beta p - \alpha)/(\beta - 1),$$

we have

$$R_{\beta, \alpha} \subset \{ w: \text{Re}\ w > \alpha \} \text{ and so } R_{\beta, \alpha} \subset S^*(\alpha).$$

Figure 1 shows this region.
For $\beta > 1$

\[
(u - \frac{\beta^2 p - \alpha}{\beta^2 - 1})^2 + \frac{\beta^2 v^2}{\beta^2 - 1} = \frac{\beta^2 (p - \alpha)^2}{(\beta^2 - 1)^2}
\]

\[\text{Figure 1}\]

ii) For $\beta = 1$, $f \in SP_p(\beta, \alpha)$, then $zf'(z)/f(z)$ belongs to the region $R_{p,\alpha}$ which contains $w = 2$ and is bounded by the parabola

\[u = \frac{(v^2 + p^2 - \alpha^2)}{2(p - \alpha)}\]

with the vertic at the point $A((p + \alpha)/2, 0)$, as shown in figure 2.

For $\beta = 1$

\[
\frac{v^2}{2(p - \alpha)} < (u - \frac{p + \alpha}{2})
\]

\[\text{Figure 2}\]
iii) For $\beta > 1$, if $f \in UCV_p(\beta, \alpha)$, then $zf''(z)/f'(z)$ lies in the region $R_{\beta, \alpha}$ such that $R_{\beta, \alpha} = \{w = u + iv \in \mathbb{C} : u > \beta \sqrt{(u - (p - 1))^2 + v^2 + \alpha - 1}, \beta \geq 0$ and $\alpha \in [0, 1]\}$, that is, part of the complex plane which contains $w = 1$ and is bounded by the ellipse

\[
(u - ((1 - \alpha) + \beta^2(p - 1))/(\beta^2 - 1))^2 + (\beta^2/(\beta^2 - 1))v^2 = \frac{\beta^2(\alpha^2 + p^2 - 2\alpha)}{2}\]

with the vertices at the points

\[
A\left((-\beta\sqrt{\alpha^2 + p^2 - 2\alpha} + (1 - \alpha) + \beta^2(p - 1))/(\beta^2 - 1), 0\right);
\]

\[
B\left((\beta\sqrt{\alpha^2 + p^2 - 2\alpha} + (1 - \alpha) + \beta^2(p - 1))/(\beta^2 - 1), 0\right);
\]

\[
C\left(((1 - \alpha) + \beta^2(p - 1))/(\beta^2 - 1), \sqrt(\alpha^2 + p^2 - 2\alpha)/(\beta^2 - 1)\right);
\]

\[
D\left(((1 - \alpha) + \beta^2(p - 1))/(\beta^2 - 1), -\sqrt(\alpha^2 + p^2 - 2\alpha)/(\beta^2 - 1)\right)
\]

Figure 3 show this region.

For $\beta > 1$

\[
(u - \frac{(1 - \alpha) + \beta^2(p - 1)}{\beta^2 - 1})^2 + \frac{\beta^2v^2}{\beta^2 - 1} = \frac{\beta^2(\alpha^2 + p^2 - 2\alpha)}{(\beta^2 - 1)^2}
\]

(iv) For $\beta = 1$, if $UCV_p(\beta, \alpha)$, then $zf''(z)/f'(z)$ belongs to the region $R_{\beta, \alpha}$ which contains $w = 2$ and is bounded by the $u = (v^2 + 2(\alpha - p) + p^2 - \alpha^2)/2(p - \alpha)$, with the vertex at the point $A((p + \alpha - 2)/2, 0)$, as shown in figure 4.
Recently, Komatu ([5], [17]) introduced a certain integral operator \( L_\lambda^\alpha (a > 0; \lambda \geq 0) \) defined by

\[
L_\lambda^\alpha f(z) = \frac{a^\lambda}{\Gamma(\lambda)} \int_0^1 t^{a-2} (\log t)^{\lambda-1} f(zt) \, dt \quad (z \in U; a > 0; \lambda \geq 0; f(z) \in A) \quad (16)
\]

Thus, if \( f(z) \in A(p) \) is of the form (1), it is easily from (16) that (see [6], [11], [14])

\[
L_\lambda^\alpha f(z) = z^p + \sum_{n=p+1}^{\infty} \frac{(a-1+p)^{\lambda}}{(a+n-1)} a_n z^n \quad (a > 0; \lambda \geq 0) \quad (17)
\]

Using the above relation, it is easy verify that

\[
z(L_\lambda^{\lambda+1} f(z))' = (a-1+p)L_\lambda^\alpha f(z) - (a-1)L_\lambda^{\lambda+1} f(z) \quad (a > 0; \lambda \geq 0) \quad (18)
\]

3− Inclusion Relationships for New Subclasses of P−valent Functions

Let \( f \) be given by (1) and the integral operator \( L_\lambda^\alpha : A(p) \to A(p) \) be given by (17). (see [8],[13])

Now, we define new classes by using the operator \( L_\lambda^\alpha \) as follows:

Definition 13.

Let \( f(z) \in A(p) \) given by (1). Then \( f(z) \in SP_{p,a}^\alpha (\beta, \alpha) \) if and only if

\[
L_\lambda^\alpha f(z) \in SP_{p}(\beta, \alpha) \quad (19)
\]
Definition 14.
Let \( f(z) \in A(p) \) given by (1). Then \( f(z) \in UC\lambda_p(\beta, \alpha) \) if and only if
\[ L_\lambda^\beta f(z) \in UC_p(\beta, \alpha) \] (20)

Definition 15.
Let \( f(z) \in A(p) \) given by (1). Then \( f(z) \in UK\lambda_p(\beta, \alpha) \) if and only if
\[ L_\lambda^\beta f(z) \in UK_p(\beta, \alpha) \] (21)

Definition 16.
Let \( f(z) \in A(p) \) given by (1). Then \( f(z) \in UQ\lambda_p(\beta, \alpha) \) if and only if
\[ L_\lambda^\beta f(z) \in UQ_p(\beta, \alpha) \] (22)

Theorem 1.
Let \( f(z) \in A(p); \alpha > 0, \beta \geq 0, \ 0 \leq \alpha < p, \lambda \geq 0. \) Then
\[ SP\lambda_{p,a}(\beta, \alpha) \subset SP\lambda_{p,a}^{\alpha+1}(\beta, \alpha). \]

Proof.
Let \( f(z) \in SP\lambda_{p,a}(\beta, \alpha) \) and suppose that
\[ \frac{z\left(t_\lambda^{\beta+1}f(z)\right)'}{L_\lambda^{\beta+1}f(z)} = p(z), \] (23)
where \( p(z) = p + p_1z + p_2z^2 + \cdots \) is analytic in \( U \) and \( p(z) \neq 0 \) for all \( z \in U. \)

Using the identity (18), we have
\[ \frac{(a - 1 + p)L_\lambda^\beta f(z)}{L_\lambda^{\beta+1}f(z)} = p(z) + (a - 1) \] (24)

By using the logarithmic differentiation on both side of (24) with respect to \( z, \) we obtain
\[ \frac{z\left(t_\lambda^{\beta+1}f(z)\right)'}{L_\lambda^{\beta+1}f(z)} = \frac{z\left(t_\lambda^{\beta+1}f(z)\right)'}{L_\lambda^{\beta+1}f(z)} + \frac{zp'(z)}{p(z) + (a - 1)} \]
which, in view of (23), yields
\[ \frac{z\left(t_\lambda^{\beta}f(z)\right)'}{L_\lambda^{\beta}f(z)} = p(z) + \frac{zp'(z)}{p(z) + (a - 1)} \] (25)

From (25), we see that
\[ Re\{h(z) + (a - 1)\} > 0, \ (z \in U), \]
and
\[ p(z) + \frac{zp'(z)}{p(z) + (a - 1)} < h(z), \]

thus, by using Lemma 1 and (23), we observe that
\[ p(z) < h(z), \]
so that
\[ f(z) \in SP\lambda_{p,a}^{\alpha+1}(\beta, \gamma), \]
this implies that
\[ SP\lambda_{p,a}(\beta, \alpha) \subset SP\lambda_{p,a}^{\alpha+1}(\beta, \alpha). \]

This completes the proof of Theorem 1.
Theorem 2.
Let \( f(z) \in A(p); a > 0, \beta \geq 0, 0 \leq \alpha < p, \lambda \geq 0 \). Then
\[
UCV_{p,a}^{\lambda} (\beta, \alpha) \subset UCV_{p,a}^{\lambda+1} (\beta, \alpha).
\]

Proof. 
\[
f(z) \in UCV_{p,a}^{\lambda} (\beta, \alpha) \iff L_{a}^{\lambda} f(z) \in UCV_{p} (\beta, \alpha)
\]
\[
\iff z \left( L_{a}^{\lambda} f(z) \right)' \in SP_{p} (\beta, \alpha)
\]
\[
\iff L_{a}^{\lambda} (zf'(z)) \in SP_{p} (\beta, \alpha)
\]
\[
\iff z f'(z) \in SP_{p}^{\lambda+1} (\beta, \alpha)
\]
\[
\iff L_{a}^{\lambda+1} (zf'(z)) \in SP_{p} (\beta, \alpha)
\]
\[
\iff z \left( L_{a}^{\lambda+1} f(z) \right)' \in SP_{p} (\beta, \alpha)
\]
\[
\iff L_{a}^{\lambda+1} f(z) \in UCV_{p} (\beta, \alpha)
\]
\[
\iff f(z) \in UCV_{p,a}^{\lambda+1} (\beta, \alpha).
\]
Which evidently proves Theorem 2.

Theorem 3.
Let \( f(z) \in A(p); a > 0, \beta \geq 0, 0 \leq \alpha < p, \lambda \geq 0 \). Then
\[
UKC_{p,a}^{\lambda} (\beta, \alpha) \subset UKC_{p,a}^{\lambda+1} (\beta, \alpha).
\]

Proof. 
Let \( f(z) \in UKC_{p,a}^{\lambda} (\beta, \alpha) \).

Then, by (21), there exists a function \( q(z) \in SP_{p} (\beta, \alpha) \) \((0 \leq \alpha < p)\) such that
\[
\frac{z \left( L_{a}^{\lambda} f(z) \right)'}{q(z)} < \psi(z) \quad \text{in } U
\]  
(26)

Taking
\[
q(z) = L_{a}^{\lambda} g(z) \in SP_{p} (\beta, \alpha).
\]  
(27)

We find from (19) and (27) that \( g(z) \in SP_{p,a}^{\lambda} (\beta, \alpha) \) and
\[
\frac{z \left( L_{a}^{\lambda} f(z) \right)'}{L_{a}^{\lambda} g(z)} < \psi(z) \quad \text{in } U.
\]

Respectively. We now let
\[
\frac{z \left( L_{a}^{\lambda+1} f(z) \right)'}{L_{a}^{\lambda+1} g(z)} = p(z).
\]  
(28)

Where \( p(z) = p + p_{1}z + p_{2}z^{2} + \cdots \)

Making use of the operator identity (18), we also have
\[
\frac{z \left( L_{a}^{\lambda} f(z) \right)'}{L_{a}^{\lambda} g(z)} = \frac{L_{a}^{\lambda} (zf'(z))}{L_{a}^{\lambda} g(z)}
\]
\[
= \frac{z \left( L_{a}^{\lambda+1} (zf'(z)) \right) + (\alpha - 1)L_{a}^{\lambda+1} (zf'(z))}{z \left( L_{a}^{\lambda+1} g(z) \right) + (\alpha - 1)L_{a}^{\lambda+1} g(z)}
\]

By Theorem 1, we know that
\[
g(z) \in SP_{p,a}^{\lambda} (\beta, \alpha)
\]
and
\[
SP_{p,a}^{\lambda} (\beta, \alpha) \subset SP_{p,a}^{\lambda+1} (\beta, \alpha)
\]  
(29)
Proof. Theorem 4. this implies that

\[ \sum_{k=1}^{\infty} a_k z^k = \frac{G(z)}{L_a^z g(z)} \]

where

\[ \text{Re}\{G(z)\} > 0, \quad (z \in U). \]

Upon substituting from (28) and (30) into (29), we have

\[ \frac{z L_a^z f(z)}{L_a^z g(z)} = \frac{z L_a^z f(z)}{L_a^z g(z)} + (a - 1) \]

\[ = \frac{z L_a^z f(z)}{L_a^z g(z)} + (a - 1) \]

Now from (28), we have

\[ z (L_a^z f(z))' = p(z) L_a^z g(z). \]

Differentiating both sides of (32) with respect to \( z \), we have

\[ \left( z (L_a^z f(z))' \right)' = L_a^z f(z) (zp'(z)) + p(z) z \]

\[ \frac{z (L_a^z f(z))'}{L_a^z g(z)} = G(z) p(z) + z p'(z). \]

Making use of (26), (31) and (33), we get

\[ \frac{z (L_a^z f(z))'}{L_a^z g(z)} = p(z) + \frac{z p'(z)}{G(z) + (a - 1)} < \psi(z) \]

since

\[ \text{Re}\{G(z) + a - 1\} > 0, \quad (z \in U). \]

Hence, be taking

\[ Q(z) = \frac{1}{G(z) + (a - 1)} \]

In (34), and applying Lemma 2, we can show that

\[ p(z) < \psi(z) \quad \text{in} \quad U \]

so that

\[ f(z) \in UKC^{(d+1)}_{p, \alpha} (\beta, \alpha) \]

this implies that

\[ UKC^{(d+1)}_{p, \alpha} (\beta, \alpha) \subset UKC^{(d+1)}_{p, \alpha} (\beta, \alpha). \]

This completes the proof of Theorem 3.

Theorem 4.

Let \( f(z) \in A(p); \ \alpha > 0, \ \beta \geq 0, \ 0 \leq \alpha < p, \ \lambda \geq 0. \) Then

\[ UQKC^{(d+1)}_{p, \alpha} (\beta, \alpha) \subset UQKC^{(d+1)}_{p, \alpha} (\beta, \alpha). \]

Proof.

\[ f(z) \in UQKC^{(d+1)}_{p, \alpha} (\beta, \alpha) \iff L_a^z f(z) \in UQKC_{p, \alpha} (\beta, \alpha) \]

\[ \iff \frac{z (L_a^z f(z))'}{L_a^z g(z)} \in UKC_{p, \alpha} (\beta, \alpha) \]

\[ \iff L_a^z (zf'(z)) \in UKC_{p, \alpha} (\beta, \alpha) \]
\[ z f'(z) \in UKC_p(\beta, \alpha) \]
\[ z f'(z) \in UKC_{p,1}^{+1}(\beta, \alpha) \]
\[ L^{\alpha+1}_\sigma(z f'(z)) \in UKC_p(\beta, \alpha) \]
\[ z \left( L^{\alpha+1}_\sigma f(z) \right)' \in UKC_p(\beta, \alpha) \]
\[ L^{\alpha+1}_\sigma f(z) \in UQC_p(\beta, \alpha) \]
\[ f(z) \in UQC_{p,1}^{+1}(\beta, \alpha). \]

Which evidently proves Theorem 4.

References:


